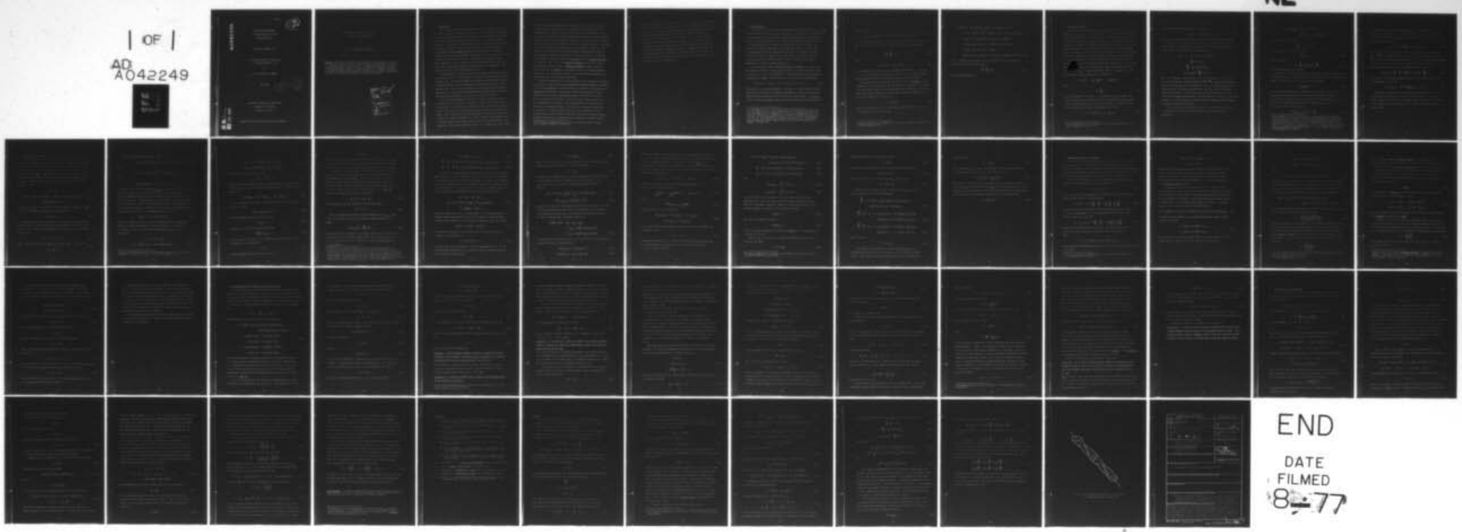
AD-A042 249 CALIFORNIA UNIV BERKELEY DEPT OF MECHANICAL ENGINEERING F/G 20/4
THE INFLUENCE OF TWIST ON THE MOTION OF STRAIGHT ELLIPTICAL JET--ETC(U)
JUN 77 D A CAULK, P M NAGHDI N00014-76-C-0474
UNCLASSIFIED UCB/AM-77-5 NL

| OF |
AD
A042249



END
DATE
FILMED
8-77

OFFICE OF NAVAL RESEARCH

Contract N00014-76-C-0474

Project NR 062-534

Report No. UCB/AM-77-5

The Influence of Twist on the Motion of Straight Elliptical Jets

by

D. A. Caulk and P. M. Naghdi

June 1977

Department of Mechanical Engineering
University of California
Berkeley, California

The Influence of Twist on the Motion

of Straight Elliptical Jets

by

D. A. Caulk and P. M. Naghdi

Abstract. This paper is concerned with a fairly detailed analysis of the motion of a straight elliptical jet of an incompressible, inviscid fluid in which the jet is allowed to twist along its axis. Our study, which includes the effects of gravity and surface tension, utilizes the nonlinear differential equations of the one-dimensional theory of a directed fluid jet. A number of theorems are proved pertaining to the motion of a twisted elliptical jet and some special solutions are obtained which illustrate the influence of twist.

1. Introduction

In the context of the three-dimensional theory, the motion of a fluid jet in air is, in general, a difficult problem. For this reason, when dealing with such fluid jet problems, it has been customary to obtain approximate solutions to the three-dimensional equations by various means. Instead of adopting a procedure of this kind, here we approach the subject from another point of view, namely via the theory of a directed (or Cosserat) curve, which is based on a one-dimensional continuum model comprising a material curve in Euclidean 3-space with two deformable directors (representing the cross-section of the jet) attached to every point of the curve. Although the application of such a direct approach to elastic rods has received considerable attention during the past decade, corresponding studies for fluid jets have been limited to a few papers by Green and Laws [5] and by Green [2,3,4].

The developments in [2,3,5] are confined to straight circular jets, while the discussion in [4] is concerned with an elliptical jet which does not twist along its axis. In the present paper, we consider a fairly detailed study of a straight elliptical jet of an incompressible fluid and allow the jet to twist along its axis. This study is of interest not only because the jet motion is more general, but also because it may indicate what conditions are necessary for more restricted motions, such as those in [2,3,4,5], to exist.

After a brief description in section 2 of certain aspects of the three-dimensional motion of the jet, in section 3 we recall the basic equations of the nonlinear theory of a directed curve in the form given by Green, Naghdi and Wenner [7] and also record some specific results concerning an incompressible medium. Next, in section 4, the basic equations of section 3 are specialized to those for a twisted elliptical jet. At this stage the field equations contain three assigned vector fields and two inertia coefficients which are still unspecified. The inertia coefficients reflect the geometry of the cross-section of the jet and the assigned vector fields involve the action of the surface tension and the external pressure, as well as gravity.[†] These quantities are identified with an appeal to certain easily accessible results in the derivation of the basic field equations for rod-like bodies from the three-dimensional equations as given by Green, Naghdi and Wenner [6]. Three-dimensional considerations of this type are used further in section 5 to provide appropriate interpretations of the kinematics of the directors and to determine the relation of the latter to the twisting of the jet and the way this twisting varies with time. As a consequence of these interpretations we define in section 5 a number of useful kinematic quantities, namely the <u>jet spin</u> ω and the <u>sectional shearing</u> γ in the jet [see Eqs. (5.5)], the <u>sectional orientation</u> θ of the jet and its rate of change, called the rate of <u>sectional rotation</u> [see Eqs. (5.11) and (5.12)], and the local <u>twist</u> per unit length of the jet [introduced following Eq. (5.12)]. Also included in section 5 is a brief discussion concerning the special case of a jet of circular cross-section.

Constitutive equations for an inviscid fluid jet are considered in section 6 and are used to complete a system of differential equations governing the motion of a straight incompressible inviscid fluid jet, referred to subsequently as an <u>ideal jet</u> for convenience. This system of differential equations is then utilized to prove several theorems pertaining to the motion of an ideal jet with and without surface tension. One of the theorems, namely Theorem 6.2, is the one-dimensional counterpart of the three-dimensional result expressing permanence of irrotational motion. Another result of particular significance is Theorem 6.4, according to which if an ideal jet is noncircular at some material point and given instant of time and if the rate of sectional rotation vanishes there, then at the material point in question the sectional rotation

[†] In the presence of gravity, the axis of the jet must be vertical in order that it remain straight as assumed here.

will have the same constant value for all time. Moreover (see Corollary 6.2), if the rate of sectional rotation vanishes at every material point and if the jet does not twist at some instant of time, then the jet must be without twist at all times during the motion. In section 7 we examine solutions of the equations for an inviscid jet in certain special cases. A theorem of correspondence is proved which relates all steady solutions for an inviscid twisting jet of circular cross-section to corresponding steady solutions in the absence of twist. In addition, we obtain a solution for the steady flow of a uniformly twisted elliptical jet and examine each of the two cases it contains. One of these reduces to a solution given by Green and Laws [5] in the special case of a circular jet.

2. The jet problem

We discuss in this section the three-dimensional motion under gravity of an incompressible fluid jet in air.† The jet is assumed to be straight and moves parallel to the gravitational field. We include in this discussion motions involving rotation about the jet axis and assume that the shape of the normal cross-section of the jet possesses symmetry with respect to two orthogonal axes in its plane. As a result of the rotational motion in the jet, the orientation of these axes may vary continuously along its length and thus give rise to a twist which, for a noncircular jet, will be apparent in the shape of the free surface. At this surface, we allow for a constant surface tension T and replace the ambient atmosphere by a constant external pressure p_o.

We use a system of fixed rectangular Cartesian coordinates $x_i = (x,y,z)$ and the associated unit base vectors $\underset{\sim}{i},\underset{\sim}{j},\underset{\sim}{k}$ such that the z-axis lies along the center line of the jet. It is also convenient to introduce another system of orthogonal coordinates $\bar{x}_i = (\bar{x},\bar{y},\bar{z})$, which are related to x_i through the transformations

$$\bar{x} = x \cos\theta + y \sin\theta \quad , \quad \bar{y} = -x \sin\theta + y \cos\theta \quad , \quad \bar{z} = z \quad , \tag{2.1}$$

where $\theta = \theta(z,t)$ is a smooth function of z and time t. At each instant of time t, equations (2.1) represent an orthogonal transformation in which the x-y plane is rotated locally an amount θ about the z-axis which remains fixed. Let $\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3$ denote the mutually orthogonal unit base vectors associated with the coordinates

---

†The term jet as it is used here denotes what is sometimes called a 'free jet' and this usage should be clearly distinguished from others in fluid mechanics. We do not consider a 'two-dimensional' jet, i.e., a two-dimensional flow in which the fluid region is bounded by two nonintersecting open cylindrical free surfaces, nor do we contemplate an immersed jet as described in Schlichting [11, Ch. 11]. The twisting jet considered here should also be distinguished from swirling flow within a rigid surface of revolution as discussed, for example, by Taylor [12].

$(\bar{x},\bar{y},\bar{z})$. Then,

$$\underset{\sim}{e}_1 = \underset{\sim}{i} \cos\theta + \underset{\sim}{j} \sin\theta \quad , \quad \underset{\sim}{e}_2 = -\underset{\sim}{i} \sin\theta + \underset{\sim}{j} \cos\theta \quad , \quad \underset{\sim}{e}_3 = \underset{\sim}{k} \quad . \tag{2.2}$$

We assume that the shape of the normal cross-section of the jet takes the form of an ellipse.* Then, the free surface of the jet may be represented by

$$\frac{\bar{x}^2}{\phi_1^2} + \frac{\bar{y}^2}{\phi_2^2} = 1 \quad , \tag{2.3}$$

where

$$\phi_1 = \phi_1(z,t) \quad , \quad \phi_2 = \phi_2(z,t) \tag{2.4}$$

are the lengths of the semiaxes of the elliptical cross-section. In the form (2.3), $\theta(z,t)$ is the local orientation of the ellipse and the three functions ϕ_1, ϕ_2 and θ are sufficient to specify the position of the free surface.

In addition to the condition that the surface (2.3) be material, on this boundary we must also require that the discontinuity in normal traction balance the surface tension and that the shearing traction be continuous. Thus, let $\underset{\sim}{t}$ denote the stress vector across a surface whose outward unit normal is $\underset{\sim}{\nu}^*$. Then, on the surface (2.3) we must have

$$\underset{\sim}{t} = (q - p_o)\underset{\sim}{\nu}^* \quad , \tag{2.5}$$

where the term q is due to the surface tension. With the help of (2.1) and (2.3), q may be expressed in the form

---

*Actually such a specific assumption is not necessary in some of the discussion of the following sections.

$$
\begin{aligned}
q = T\{[&\theta_z \sin\chi \cos\chi(\phi_2^2 - \phi_1^2) - (\phi_1\phi_{2z} \sin^2\chi + \phi_2\phi_{1z} \cos^2\chi)]^2 \\
&+ \phi_1^2 \sin^2\chi + \phi_2^2 \cos^2\chi\}^{-\frac{3}{2}} \cdot \{(\phi_1^2 \sin^2\chi + \phi_2^2 \cos^2\chi)(\phi_{1zz}\phi_2 \cos^2\chi \\
&+ \phi_{2zz}\phi_1 \sin^2\chi - [\theta_z(\phi_2^2 - \phi_1^2)]_z \sin\chi \cos\chi - \phi_1\phi_2\theta_z^2) \\
&- 2[(\phi_1\phi_{2z} - \phi_2\phi_{1z})\sin\chi \cos\chi - \theta_z(\phi_2^2 \cos^2\chi + \phi_1^2 \sin^2\chi)] \\
&\times [(\phi_2\phi_{2z} - \phi_1\phi_{1z})\sin\chi \cos\chi + \phi_1\phi_2\theta_z] \\
&- \phi_1\phi_2[(\phi_{1z} \cos\chi - \phi_2\theta_z \sin\chi)^2 + (\phi_{2z} \sin\chi + \phi_1\theta_z \cos\chi)^2 + 1]\}\ , \qquad (2.6)
\end{aligned}
$$

where a subscript denotes partial differentiation, χ is the polar angle in the plane of the cross-section given by

$$\frac{\bar{y}}{\bar{x}} = \frac{\phi_2}{\phi_1} \tan\chi \qquad (2.7)$$

and $\bar{x}$ and $\bar{y}$ satisfy (2.3).

3. A directed fluid jet

We summarize in this section the basic equations governing the motion of an incompressible fluid jet. These are derived by a direct approach based on the concept of a directed (or a Cosserat) curve. A continuum of this kind, hereafter referred to as $\mathcal{R}$, is an oriented space curve to every point of which two directors are attached. We confine attention to a purely mechanical theory contained in the more general thermodynamical development of Green, Naghdi and Wenner [7]. Many of the results of this section parallel those of a related work by Green [3], although the present kinematics are somewhat less restrictive.‡

Let the particles (or the material points) of the material line of the directed curve $\mathcal{R}$ be identified with the convected coordinate ξ. In the present configuration at time t, let the space curve occupied by the material line of $\mathcal{R}$ be referred to by c, let $\mathbf{r}$ be the position vector of c and $\mathbf{d}_\alpha$ $(\alpha = 1,2)$ the directors at $\mathbf{r}$. Then, a motion of $\mathcal{R}$ is specified by

$$\mathbf{r} = \mathbf{r}(\xi,t) \quad , \quad \mathbf{d}_\alpha = \mathbf{d}_\alpha(\xi,t) \quad , \quad [\mathbf{d}_1\mathbf{d}_2\mathbf{a}_3] \neq 0 \quad , \tag{3.1}$$

where

$$\mathbf{a}_3 = \frac{\partial \mathbf{r}}{\partial \xi} \tag{3.2}$$

is a vector tangent to the curve c. The condition $(3.1)_3$ will ensure that neither of the directors is tangent to the curve and that $\mathbf{d}_1$ and $\mathbf{d}_2$ maintain the same relative orientation to each other and to $\mathbf{a}_3$. We denote the element of arc length along the curve by ds, and from (3.2) we have

$$ds = (a_{33})^{\frac{1}{2}}\, d\xi \quad , \quad a_{33} = \mathbf{a}_3 \cdot \mathbf{a}_3 \quad . \tag{3.3}$$

---

‡Our kinematics allow for the rotation of the director pair relative to the tangent vector $\mathbf{a}_3$ defined by (3.2).

The velocity and director velocities are defined by

$$\underset{\sim}{v} = \dot{\underset{\sim}{r}}(\xi,t) \quad , \quad \underset{\sim}{w}_{\alpha} = \dot{\underset{\sim}{d}}_{\alpha}(\xi,t) \quad , \tag{3.4}$$

where a superposed dot designates the material time derivative holding ξ fixed.

A summary of the conservation laws in the purely mechanical theory of a directed curve in integral form is contained in [7]. For our present purpose, it will suffice to record them in local form, namely

$$\begin{gathered}
\rho(a_{33})^{\frac{1}{2}} = \lambda \quad \text{or} \quad \dot{\rho} + \rho(a_{33})^{-1}\, \underset{\sim}{a}_3 \cdot \frac{\partial \underset{\sim}{v}}{\partial \xi} = 0 \quad , \\
\frac{\partial \underset{\sim}{n}}{\partial \xi} + \lambda \underset{\sim}{f} = \lambda \dot{\underset{\sim}{v}} \quad , \\
\frac{\partial \underset{\sim}{p}^{\alpha}}{\partial \xi} + \lambda \underset{\sim}{\ell}^{\alpha} = \underset{\sim}{\pi}^{\alpha} + \lambda y^{\alpha\beta}\, \dot{\underset{\sim}{w}}_{\beta} \quad , \\
\underset{\sim}{a}_3 \times \underset{\sim}{n} + \underset{\sim}{d}_{\alpha} \times \underset{\sim}{\pi}^{\alpha} + \frac{\partial \underset{\sim}{d}_{\alpha}}{\partial \xi} \times \underset{\sim}{p}^{\alpha} = \underset{\sim}{0} \quad ,
\end{gathered} \tag{3.5}$$

where λ is a function of ξ only, ρ is the mass density (per unit length) of the curve and the symmetric inertia coefficients $y^{\alpha\beta} = y^{\alpha\beta}(\xi)$ are associated with the director velocities $\underset{\sim}{w}_{\alpha}$ and are independent of time. In addition, $\underset{\sim}{n}$ denotes the contact force, $\underset{\sim}{p}^{\alpha}$ the contact director forces and $\underset{\sim}{\pi}^{\alpha}$ the intrinsic director forces per unit length of c; the vector fields $\underset{\sim}{f}$ and $\underset{\sim}{\ell}^{\alpha}$ represent the assigned force and the assigned director forces, each per unit mass of the curve. The field equations (3.5) are consequences of balance of mass, momentum, director momentum and moment of momentum, respectively.

For an arbitrary part $\xi_1 \leq \xi \leq \xi_2$ of the curve c of $\mathcal{R}$, the rate of work by all external forces minus the rate of increase of kinetic energy may be expressed as

$$[\underset{\sim}{n} \cdot \underset{\sim}{v} + \underset{\sim}{p}^{\alpha} \cdot \underset{\sim}{w}_{\alpha}]_{\xi_1}^{\xi_2} + \int_{\xi_1}^{\xi_2} \rho(\underset{\sim}{f} \cdot \underset{\sim}{v} + \underset{\sim}{\ell}^{\alpha} \cdot \underset{\sim}{w}_{\alpha})ds$$

$$- \frac{d}{dt} \int_{\xi_1}^{\xi_2} \tfrac{1}{2}\, \rho(\underset{\sim}{v} \cdot \underset{\sim}{v} + y^{\alpha\beta}\, \underset{\sim}{w}_{\alpha} \cdot \underset{\sim}{w}_{\beta})ds \quad , \tag{3.6}$$

where we have used the notation

$$[f]_{\xi_1}^{\xi_2} = f(\xi_2) - f(\xi_1) \quad . \tag{3.7}$$

With the help of (3.5), (3.6) reduces to

$$\int_{\xi_1}^{\xi_2} \rho\, P\, ds \quad , \tag{3.8}$$

where P defined by

$$\lambda P = \underset{\sim}{n} \cdot \frac{\partial \underset{\sim}{v}}{\partial \xi} + \underset{\sim}{\pi}^{\alpha} \cdot \underset{\sim}{w}_{\alpha} + \underset{\sim}{p}^{\alpha} \cdot \frac{\partial \underset{\sim}{w}_{\alpha}}{\partial \xi} \tag{3.9}$$

is the mechanical power per unit mass.

From now on we restrict attention to a directed curve which is homogeneous and incompressible. In order to reflect the latter property in the one-dimensional theory, as in the work of Green [3], we adopt the condition

$$\frac{d}{dt}\, [\underset{\sim}{d}_1 \underset{\sim}{d}_2 \underset{\sim}{a}_3] = 0 \quad . \tag{3.10}$$

Strictly speaking, two additional conditions are necessary in order to fully characterize incompressibility in the context of a general directed curve. However, for the limited scope of the present paper, the condition (3.10) is sufficient.* By performing the indicated differentiation and using (3.4), (3.10) may be written alternatively as

$$e^{\alpha\beta} \underset{\sim}{d}_{\beta} \times \underset{\sim}{a}_3 \cdot \underset{\sim}{w}_{\alpha} + \underset{\sim}{d}_1 \times \underset{\sim}{d}_2 \cdot \frac{\partial \underset{\sim}{v}}{\partial \xi} = 0 \quad , \tag{3.11}$$

---

*In an appendix at the end of the paper, a 1-1 correspondence is established between the field equations (3.5) of the direct theory and the corresponding equations which emerge from the three-dimensional equations when in the three-dimensional theory the position vector is approximated by the form $(A14)_1$. Keeping this in mind, the specification of (3.10) is motivated from an examination of the condition of incompressibility in the three-dimensional theory when the position vector is approximated by $(A14)_1$. See, in this connection, the last paragraph of the Appendix.

where $e^{\alpha\beta}$ is the two-dimensional permutation symbol. To complete the theory of a directed curve under the constraint condition (3.10) or (3.11), we assume that each of the functions $\underset{\sim}{n}, \underset{\sim}{\pi}^{\alpha}$ and $\underset{\sim}{p}^{\alpha}$ is determined to within an additive constraint response so that

$$\underset{\sim}{n} = \underset{\sim}{\bar{n}} + \underset{\sim}{\hat{n}} \quad , \quad \underset{\sim}{\pi}^{\alpha} = \underset{\sim}{\bar{\pi}}^{\alpha} + \underset{\sim}{\hat{\pi}}^{\alpha} \quad , \quad \underset{\sim}{p}^{\alpha} = \underset{\sim}{\bar{p}}^{\alpha} + \underset{\sim}{\hat{p}}^{\alpha} \quad , \tag{3.12}$$

where $\underset{\sim}{\hat{n}}$, $\underset{\sim}{\hat{\pi}}^{\alpha}$ and $\underset{\sim}{\hat{p}}^{\alpha}$ are determined by constitutive equations and the functions $\underset{\sim}{\bar{n}}(\xi,t)$, $\underset{\sim}{\bar{\pi}}^{\alpha}(\xi,t)$ and $\underset{\sim}{\bar{p}}^{\alpha}(\xi,t)$ are the response due to the constraint; the latter quantities are arbitrary functions of ξ,t and do no work. Introducing the Lagrange multiplier $\bar{p}$ and with reference to the expression (3.9) for the mechanical power, the constraint response functions $\underset{\sim}{\bar{n}}$, $\underset{\sim}{\bar{\pi}}^{\alpha}$ and $\underset{\sim}{\bar{p}}^{\alpha}$ must satisfy

$$(\underset{\sim}{\bar{n}} + \bar{p}\underset{\sim}{d}_1 \times \underset{\sim}{d}_2) \cdot \frac{\partial \underset{\sim}{v}}{\partial \xi} + (\underset{\sim}{\bar{\pi}}^{\alpha} + \bar{p}e^{\alpha\beta}\underset{\sim}{d}_{\beta} \times \underset{\sim}{a}_3) \cdot \underset{\sim}{w}_{\alpha} + \underset{\sim}{\bar{p}}^{\alpha} \cdot \frac{\partial \underset{\sim}{w}_{\alpha}}{\partial \xi} = 0 \tag{3.13}$$

for all values of $\partial \underset{\sim}{v}/\partial\xi$, $\underset{\sim}{w}_{\alpha}$ and $\partial \underset{\sim}{w}_{\alpha}/\partial\xi$ subject to the constraint (3.10). Since $\bar{p}$ is arbitrary, it follows that

$$\underset{\sim}{\bar{n}} = -\bar{p}\underset{\sim}{d}_1 \times \underset{\sim}{d}_2 \quad , \quad \underset{\sim}{\bar{\pi}}^{\alpha} = -\bar{p}e^{\alpha\beta}\underset{\sim}{d}_{\beta} \times \underset{\sim}{a}_3 \quad , \quad \underset{\sim}{\bar{p}}^{\alpha} = \underset{\sim}{0} \quad . \tag{3.14}$$

Apart from the constraint (3.10), the preceding equations are fully general within the context of the theory of a directed curve. In the next section, however, we specialize the foregoing results to the case of a straight jet and consider a class of motions which is of particular interest in the present paper.

4. A straight jet with twist.

In terms of the fixed system of coordinates (x,y,z) introduced in section 2, we now consider motions of a directed curve in which the material line of $\mathcal{R}$ is identified with the z-axis and the director pair lies in the x-y plane. For later convenience, however, we first consider motions of $\mathcal{R}$ at time $\tau \leq t$. Thus, let $\underset{\sim}{r}(\tau) = \underset{\sim}{r}(\xi,\tau)$ and $\underset{\sim}{d}_\alpha(\tau) = \underset{\sim}{d}_\alpha(\xi,\tau)$ designate at time τ the position vector of the material line of $\mathcal{R}$ and the directors, respectively; and, with reference to the present configuration of $\mathcal{R}$ at time t, we adopt the notations $\underset{\sim}{r} = \underset{\sim}{r}(t)$ and $\underset{\sim}{d}_\alpha = \underset{\sim}{d}_\alpha(t)$. Then,

$$\underset{\sim}{r}(\tau) = z\underset{\sim}{k} \quad , \tag{4.1}$$

$$\underset{\sim}{d}_1(\tau) = \phi_1(\underset{\sim}{i} \cos \psi_1 + \underset{\sim}{j} \sin \psi_1) \quad , \quad \underset{\sim}{d}_2(\tau) = \phi_2(\underset{\sim}{i} \cos \psi_2 + \underset{\sim}{j} \sin \psi_2) \quad ,$$
$$[\underset{\sim}{d}_1(\tau)\underset{\sim}{d}_2(\tau)\underset{\sim}{a}_3(\tau)] \neq 0 \quad , \tag{4.2}$$

where z, ϕ_α and ψ_α are each functions of ξ and τ and $\underset{\sim}{a}_3(\tau) = \partial\underset{\sim}{r}(\tau)/\partial\xi$. In view of $(4.2)_3$, the functions ψ_1 and ψ_2 are so restricted that

$$\sin(\psi_2 - \psi_1) \neq 0 \quad . \tag{4.3}$$

It is clear from (4.2) that the directors lie in the plane of the normal cross-section of the jet, but they are not necessarily orthogonal. From (3.4), (4.1) and (4.2), the velocity, the director velocities and the director accelerations at time τ are

$$\underset{\sim}{v}(\tau) = v\underset{\sim}{k} \quad , \quad \underset{\sim}{v} = \underset{\sim}{v}(t) \quad , \tag{4.4}$$

$$\underset{\sim}{w}_\alpha(\tau) = w_\alpha(\underset{\sim}{i} \cos \psi_\alpha + \underset{\sim}{j} \sin \psi_\alpha) + \omega_\alpha \phi_\alpha(-\underset{\sim}{i} \sin \psi_\alpha + \underset{\sim}{j} \cos \psi_\alpha) \qquad (\text{no sum on } \alpha) \quad ,$$
$$\underset{\sim}{w}_\alpha = \underset{\sim}{w}_\alpha(t) \quad , \tag{4.5}$$

$$\dot{\underset{\sim}{w}}_\alpha(\tau) = \phi_\alpha(\dot{\zeta}_\alpha + \zeta_\alpha^2 - \omega_\alpha^2)(\underset{\sim}{i} \cos \psi_\alpha + \underset{\sim}{j} \sin \psi_\alpha)$$

$$+ \phi_\alpha(2\zeta_\alpha\omega_\alpha + \dot{\omega}_\alpha)(-\underset{\sim}{i} \sin \psi_\alpha + \underset{\sim}{j} \cos \psi_\alpha) \quad \text{(no sum on } \alpha) \quad , \qquad (4.6)$$

where

$$v(\tau) = \dot{z}(\tau) \quad , \quad w_\alpha(\tau) = \dot{\phi}_\alpha(\tau) \quad , \quad \omega_\alpha(\tau) = \dot{\psi}_\alpha(\tau) \qquad (4.7)$$

and

$$w_\alpha(\tau) = \phi_\alpha(\tau)\zeta_\alpha(\tau) \quad \text{(no sum on } \alpha) \quad . \qquad (4.8)$$

It should be clear that the functions $v, w_\alpha, \omega_\alpha, \psi_\alpha$ and ζ_α in (4.4) to (4.6) depend on ξ and τ, although this dependence has not been explicitly displayed. Also, the superposed dots in (4.6) to (4.8) stand for the material time derivative with respect to τ. Now, without loss in generality, we may fix the orientation of the directors relative to the material line of $\mathcal{R}$ in one configuration. In the present development, we make this choice relative to the present configuration and take the directors to be orthogonal at time t, i.e.,[†]

$$\psi_1(\xi,t) = \theta \quad , \quad \psi_2(\xi,t) = \theta + \frac{\pi}{2} \quad , \qquad (4.9)$$

where θ is a function of ξ and t. We leave θ unspecified for the moment and return to it later in this section. Using (4.9) and (2.2), the directors, the director velocities and the director accelerations at time t can be expressed as

$$\underset{\sim}{d}_1 = \phi_1 \underset{\sim}{e}_1 \quad , \quad \underset{\sim}{d}_2 = \phi_2 \underset{\sim}{e}_2 \quad , \qquad (4.10)$$

$$\underset{\sim}{w}_1 = \phi_1(\zeta_1 \underset{\sim}{e}_1 + \omega_1 \underset{\sim}{e}_2) \quad , \quad \underset{\sim}{w}_2 = \phi_2(\zeta_2 \underset{\sim}{e}_2 - \omega_2 \underset{\sim}{e}_1) \quad , \qquad (4.11)$$

[†]It should be noted that this specification is made after calculating the expressions for the director velocities $(4.5)_1$ and accelerations (4.6).

and

$$\dot{\underset{\sim}{w}}_1 = (\dot{\zeta}_1 + \zeta_1^2 - \omega_1^2)\phi_1 \underset{\sim}{e}_1 + (2\zeta_1\omega_1 + \dot{\omega}_1)\phi_1 \underset{\sim}{e}_2 \quad ,$$
$$\dot{\underset{\sim}{w}}_2 = (\dot{\zeta}_2 + \zeta_2^2 - \omega_2^2)\phi_2 \underset{\sim}{e}_2 - (2\zeta_2\omega_2 + \dot{\omega}_2)\phi_2 \underset{\sim}{e}_1 \quad . \tag{4.12}$$

Also, by (3.2), (3.3) and (4.1), we have

$$\underset{\sim}{a}_3 = z'\underset{\sim}{k} \quad , \quad (a_{33})^{\frac{1}{2}} = z' \quad , \tag{4.13}$$

where a prime denotes partial differentiation with respect to ξ. With the help of (4.10) and $(4.13)_1$, the nonvanishing constraint response functions (3.14) reduce to

$$\bar{\underset{\sim}{n}} = -\bar{p}\phi_1\phi_2 \underset{\sim}{e}_3 \quad , \quad \bar{\underset{\sim}{\pi}}^1 = -\bar{p}\phi_2 z' \underset{\sim}{e}_1 \quad , \quad \bar{\underset{\sim}{\pi}}^2 = -\bar{p}\phi_1 z' \underset{\sim}{e}_2 \quad . \tag{4.14}$$

The incompressibility condition (3.11), *in view of* (4.10), (4.11), (4.4) and (4.13), becomes

$$z'\phi_2 w_1 + z'\phi_1 w_2 + \phi_1\phi_2 v' = 0 \tag{4.15}$$

and from combination of (4.15) and (4.8) we have

$$\zeta_1 + \zeta_2 + v_z = 0 \quad . \tag{4.16}$$

Alternatively, (4.15) can be written in the form

$$\dot{\overline{(\phi_1\phi_2)}} + \phi_1\phi_2 v_z = 0 \quad . \tag{4.17}$$

Now, in terms of (4.4) and (4.13), the differential equation $(3.5)_1$ for the mass density ρ can be expressed as

$$\dot{\rho} + \rho v_z = 0 \tag{4.18}$$

and this along with (4.17) results in the solution

$$\rho = \beta(\xi)\phi_1\phi_2 \quad , \tag{4.19}$$

where β is an arbitrary function of ξ only. Since we have assumed the medium to be homogeneous, β must be a constant; and thus, aside from a factor β, the constraint (4.15) permits us to identify the density ρ of the jet with $\phi_1\phi_2$.

In addition to the special choices (4.1) and (4.2), we further restrict attention to a jet in which the shape of the normal cross-section possesses two perpendicular axes of symmetry.* If we choose θ in (4.9) such that it measures the angle between one of these axes and the x-axis, we may then characterize this geometric symmetry through the requirement that at time t the kinetic energy per unit mass due to the directors, i.e., $\frac{1}{2}y^{\alpha\beta}\underset{\sim}{w}_\alpha \cdot \underset{\sim}{w}_\beta$, remains invariant under the separate transformations

$$\underset{\sim}{d}_1 \to -\underset{\sim}{d}_1 \quad \text{or} \quad \underset{\sim}{d}_2 \to -\underset{\sim}{d}_2 \quad . \tag{4.20}$$

Hence, we conclude that the coefficients y^{12}, y^{21} must vanish:

$$y^{12} = y^{21} = 0 \quad . \tag{4.21}$$

Next, we examine the reduction of the field equations $(3.5)_{2,3,4}$. With (3.12), (4.10), (4.13) and (4.14), the moment of momentum equation $(3.5)_4$ becomes

$$\underset{\sim}{a}_3 \times \hat{\underset{\sim}{n}} + \underset{\sim}{d}_\alpha \times \hat{\underset{\sim}{\pi}}^\alpha + \frac{\partial \underset{\sim}{d}_\alpha}{\partial \xi} \times \hat{\underset{\sim}{p}}^\alpha = \underset{\sim}{0} \quad . \tag{4.22}$$

The expression (4.22) is regarded as an identity which places restrictions on the form of the response functions $\hat{\underset{\sim}{n}}, \hat{\underset{\sim}{\pi}}^\alpha$ and $\hat{\underset{\sim}{p}}^\alpha$. With the help of (3.12), (4.4), (4.10) to (4.14) and (4.21), the momentum and director momentum equations $(3.5)_{2,3}$ reduce to

*Note that this is a purely geometric restriction and in no way limits the generality of any material symmetry that may be present in the jet. In line with the remark made in the footnote following Eq. (2.2), it should be noted that at this stage in the development by the direct approach, the assumption that the cross-section is an ellipse has not been explicitly utilized. In fact, the conclusion (4.21) is not limited to elliptical jets and the assumption that the cross-section is an ellipse is actually introduced in the direct theory by (4.28).

$$\hat{\mathbf{n}}' - (\phi_1\phi_2\bar{p})'\mathbf{e}_3 + \lambda\mathbf{f} = \lambda\dot{v}\mathbf{e}_3 \quad , \tag{4.23}$$

$$\hat{\mathbf{p}}^{1\prime} + \lambda\boldsymbol{\ell}^1 = \hat{\boldsymbol{\pi}}^1 - \bar{p}\phi_2 z'\mathbf{e}_1 + \lambda y^{11}\phi_1\{(\dot{\zeta}_1 + \zeta_1^2 - \omega_1^2)\mathbf{e}_1 + (2\zeta_1\omega_1 + \dot{\omega}_1)\mathbf{e}_2\} \quad , \tag{4.24}$$

$$\hat{\mathbf{p}}^{2\prime} + \lambda\boldsymbol{\ell}^2 = \hat{\boldsymbol{\pi}}^2 - \bar{p}\phi_1 z'\mathbf{e}_2 + \lambda y^{22}\phi_2\{(\dot{\zeta}_2 + \zeta_2^2 - \omega_2^2)\mathbf{e}_2 - (2\zeta_2\omega_2 + \dot{\omega}_2)\mathbf{e}_1\} \quad . \tag{4.25}$$

In order to specify appropriate forms for the assigned vector fields $\mathbf{f}$ and $\boldsymbol{\ell}^\alpha$, as well as the inertia coefficients y^{11} and y^{22}, we now make use of various three-dimensional results recorded in the Appendix. In terms of the convected (Lagrangian) coordinates θ^i $(i = 1,2,3)$ defined over the three-dimensional region of space occupied by the fluid, as well as the kinematic assumption $(A14)_1$, the base vectors $\mathbf{g}_i$ and the determinant of the metric tensor g are given by

$$\begin{gathered}
\mathbf{g}_1 = \phi_1\mathbf{e}_1 \quad , \quad \mathbf{g}_2 = \phi_2\mathbf{e}_2 \quad , \\
\mathbf{g}_3 = z'\{\mathbf{e}_3 + \theta^1(\phi_{1z}\mathbf{e}_1 + \phi_1\theta_z\mathbf{e}_2) + \theta^2(\phi_{2z}\mathbf{e}_2 - \phi_2\theta_z\mathbf{e}_1)\} \quad , \\
g^{\frac{1}{2}} = [\mathbf{g}_1\mathbf{g}_2\mathbf{g}_3] = z'\phi_1\phi_2 \quad ,
\end{gathered} \tag{4.26}$$

where we have made use of (4.1), (4.10) and (A1). To be consistent with the geometric symmetry characterized by (4.20), the function F, which determines the lateral free boundary of the jet through the expression (A7), must satisfy

$$F(\theta^1,\theta^2) = F(-\theta^1,\theta^2) = F(\theta^1,-\theta^2) \quad . \tag{4.27}$$

Restricting attention to the case in which the cross-section of the jet is elliptical and specifying F by

$$F = (\theta^1)^2 + (\theta^2)^2 - 1 \quad , \tag{4.28}$$

it is seen from (4.10) and (A14) that (4.28) is equivalent to (2.3). It then follows from the combination of (2.5) and (A4) that at the surface (4.28) the vector fields $\mathbf{T}^i$ assume the form

$$\underset{\sim}{T}^i = (q-p_o)g^{\frac{1}{2}}\underset{\sim}{g}^i \quad , \tag{4.29}$$

where q is given by (2.6). Also, since the axis of the jet is parallel to the gravitational field, the body force vector $\underset{\sim}{f}^*$ is given by

$$\underset{\sim}{f}^* = -g^*\underset{\sim}{e}_3 \quad , \tag{4.30}$$

where we use the temporary notation g^* for the gravitational constant in order to avoid confusion with the determinant of the metric tensor g. Using $(4.26)_4$, (4.29), (4.30) and (A12), the expressions (A19) and (A20) reduce to

$$\lambda\underset{\sim}{f} = -\rho^* z'\phi_1\phi_2 g^*\underset{\sim}{e}_3\int_{\mathcal{a}} d\theta^1 d\theta^2 - \int_{\partial\mathcal{a}}(p_o-q)g^{\frac{1}{2}}(\underset{\sim}{g}^1 d\theta^2 - \underset{\sim}{g}^2 d\theta^1) \quad , \tag{4.31}$$

$$\lambda\underset{\sim}{\ell}^\alpha = -\int_{\partial\mathcal{a}}(p_o-q)g^{\frac{1}{2}}\theta^\alpha(\underset{\sim}{g}^1 d\theta^2 - \underset{\sim}{g}^2 d\theta^1) \quad , \tag{4.32}$$

where ρ^* is the constant (three-dimensional) density of the fluid. The region of integration $\mathcal{a}$ bounded by $\partial\mathcal{a}$ is defined in the Appendix. In the present case, $\partial\mathcal{a}$ denotes the closed curve (4.28) and $\mathcal{a}$ is the section of a surface $\theta^3 = \text{const.}$ enclosed by $\partial\mathcal{a}$. In addition, from (4.26)

$$\begin{aligned} g^{\frac{1}{2}}(\underset{\sim}{g}^1 d\theta^2 - \underset{\sim}{g}^2 d\theta^1) &= -\underset{\sim}{g}_3 \times [\underset{\sim}{g}_1 d\theta^1 - \underset{\sim}{g}_2 d\theta^2] \\ &= -z'[\phi_1\underset{\sim}{e}_2 - (\theta_z\phi_1^2\theta^1 + \phi_1\phi_{2z}\theta^2)\underset{\sim}{e}_3]d\theta^1 \\ &\quad + z'[\phi_2\underset{\sim}{e}_1 + (\theta_z\phi_2^2\theta^2 - \phi_2\phi_{1z}\theta^1)\underset{\sim}{e}_3]d\theta^2 . \end{aligned} \tag{4.33}$$

An examination of the expression (2.6) for q reveals that it possesses symmetry properties that may be characterized by

$$q(\phi_1,\phi_2,\theta_z,\chi) = q(\phi_1,\phi_2,\theta_z,\chi+\pi) \quad , \tag{4.34}$$

$$q(\phi_1,\phi_2,\theta_z,\chi) = q(\phi_2,\phi_1,\theta_z,\chi+\tfrac{\pi}{2}) \quad . \tag{4.35}$$

Actually the function q, in addition to χ, ϕ_α and θ_z, depends also on $\phi_{\alpha z}$ and $\phi_{\alpha zz}$. Although the latter dependence is not explicitly displayed in (4.34) and (4.35), the argument ϕ_α is taken to stand for the triple $(\phi_\alpha, \phi_{\alpha z}, \phi_{\alpha zz})$. We adopt the same notation for the arguments of the functions $h^{\alpha\beta}$ introduced below. Keeping the symmetry (4.34) and (4.35) in mind, we first parameterize θ^1 and θ^2 on the closed curve (4.28) by

$$\theta^1 = \cos\chi \quad , \quad \theta^2 = \sin\chi \quad , \quad 0 \leqq \chi \leqq 2\pi \quad . \tag{4.36}$$

In view of (4.34) and the symmetry of the trigonometric functions in (4.36), we then easily conclude that

$$\int_{\partial\mathcal{a}} q\,d\theta^\alpha = 0 \quad , \quad \int_{\partial\mathcal{a}} q\,\theta^\alpha\theta^\beta d\theta^\gamma = 0 \qquad (\alpha,\beta,\gamma = 1,2). \tag{4.37}$$

Now put

$$h^{\alpha\beta}(\phi_1,\phi_2,\theta_z) = \int_{\partial\mathcal{a}} q\,\theta^\alpha d\theta^\beta \tag{4.38}$$

and observe from (4.35) and (4.36) that

$$\begin{aligned} h^{11}(\phi_2,\phi_1,\theta_z) = h^{22}(\phi_1,\phi_2,\theta_z) &= -\,h^{11}(\phi_1,\phi_2,\theta_z) \quad , \\ h^{12}(\phi_1,\phi_2,\theta_z) &= h^{21}(\phi_2,\phi_1,\theta_z) \quad . \end{aligned} \tag{4.39}$$

Hence, only two of the four functions defined by (4.38) are independent. For convenience, we set

$$h = h^{12} \quad , \quad m = h^{11} \tag{4.40}$$

and in what follows express all integrals of the type (4.38) in terms of h and m. After performing the integration in (4.31) and (4.32) with the help of (4.33) to (4.40), we obtain

$$\lambda f = z'\{-\rho^*\pi\phi_1\phi_2 g^* + \pi p_o(\phi_1\phi_2)_z - \phi_2\phi_{1z}h(\phi_1,\phi_2,\theta_z)$$

$$-\phi_1\phi_{2z}h(\phi_2,\phi_1,\theta_z) + \theta_z(\phi_1^2-\phi_2^2)m(\phi_1,\phi_2,\theta_z)\}e_3 \quad , \tag{4.41}$$

$$\lambda \underset{\sim}{\ell}^1 = z'\phi_2[-\pi p_o + h(\phi_1,\phi_2,\theta_z)]\underset{\sim}{e}_1 - z'\phi_1 m(\phi_1,\phi_2,\theta_z)\underset{\sim}{e}_2 \quad , \tag{4.42}$$

$$\lambda \underset{\sim}{\ell}^2 = z'\phi_1[-\pi p_o + h(\phi_2,\phi_1,\theta_z)]\underset{\sim}{e}_2 + z'\phi_2 m(\phi_2,\phi_1,\theta_z)\underset{\sim}{e}_1 \quad , \tag{4.43}$$

where

$$h(\phi_1,\phi_2,\theta_z) = \int_0^{2\pi} q\cos^2\chi\, d\chi \quad , \tag{4.44}$$

$$m(\phi_1,\phi_2,\theta_z) = -\tfrac{1}{2}\int_0^{2\pi} q\sin 2\chi\, d\chi \quad , \tag{4.45}$$

in view of (4.36), (4.38) and (4.40). We do not record here the explicit expressions for h and m, since they will not be needed in their most general form in the subsequent development. However, it may be noted that due to the additional symmetry of q when $\theta_z = 0$ the function m satisfies

$$m(\phi_1,\phi_2,0) = 0 \quad . \tag{4.46}$$

Also, by $(4.39)_1$ and (4.40), we have

$$m(\phi_1,\phi_1,\theta_z) = 0 \tag{4.47}$$

so that m vanishes identically in the absence of <u>twist</u> ($\theta_z = 0$) or in the case of a circular jet* ($\phi_1 = \phi_2$).

Next, we use (A10) and (A21) to determine λ and $y^{\alpha\beta}$. From (A10), (4.26) and (4.28), we obtain

$$\lambda = \rho^*\pi z'\phi_1\phi_2 \tag{4.48}$$

---

*The notion of <u>twist</u> and the conditions under which the jet is circular will be made more precise in the next section.

which, together with $(3.5)_1$ and $(4.13)_2$, yields

$$\rho = \rho^* \pi \phi_1 \phi_2 \quad . \tag{4.49}$$

Since ρ^* is constant, comparison of (4.48) and (4.19) suggests that we put

$$\beta = \rho^* \pi = \text{const.} \quad . \tag{4.50}$$

Using (4.48) in (A21), we again obtain $(4.21)_{1,2}$ and the values

$$y^{11} = y^{22} = \frac{1}{4} \quad . \tag{4.51}$$

Finally, if we use (4.41) to (4.43), (4.48), (4.21) and (4.51) in the field equations (4.23) to (4.25), they become

$$\begin{aligned} \frac{\partial \hat{\underset{\sim}{n}}}{\partial z} = \{ & p_z + \pi\rho^* \phi_1 \phi_2 g^* + \phi_2 \phi_{1z} h(\phi_1, \phi_2, \theta_z) + \phi_1 \phi_{2z} h(\phi_2, \phi_1, \theta_z) \\ & + \theta_z (\phi_2^2 - \phi_1^2) m(\phi_1, \phi_2, \theta_z) + \pi\rho^* \phi_1 \phi_2 \dot{v} \} \underset{\sim}{e}_3 \quad , \end{aligned} \tag{4.52}$$

$$\begin{aligned} \left(\frac{\partial \hat{\underset{\sim}{p}}^1}{\partial z} - \hat{\underset{\sim}{\pi}}^1 / z'\right)\phi_1 = & \{ -p - \phi_1 \phi_2 h(\phi_1, \phi_2, \theta_z) + \frac{1}{4} \pi\rho^* \phi_1^3 \phi_2 (\dot{\zeta}_1 + \zeta_1^2 - \omega_1^2) \} \underset{\sim}{e}_1 \\ & + \{ \phi_1^2 m(\phi_1, \phi_2, \theta_z) + \frac{1}{4} \pi\rho^* \phi_1^3 \phi_2 (\dot{\omega}_1 + 2\zeta_1 \omega_1) \} \underset{\sim}{e}_2 \quad , \end{aligned} \tag{4.53}$$

$$\begin{aligned} \left(\frac{\partial \hat{\underset{\sim}{p}}^2}{\partial z} - \hat{\underset{\sim}{\pi}}^2 / z'\right)\phi_2 = & \{ -p - \phi_1 \phi_2 h(\phi_2, \phi_1, \theta_z) + \frac{1}{4} \pi\rho^* \phi_2^3 \phi_1 (\dot{\zeta}_2 + \zeta_2^2 - \omega_2^2) \} \underset{\sim}{e}_2 \\ & - \{ \phi_2^2 m(\phi_2, \phi_1, \theta_z) + \frac{1}{4} \pi\rho^* \phi_2^3 \phi_1 (\dot{\omega}_2 + 2\zeta_2 \omega_2) \} \underset{\sim}{e}_1 \quad , \end{aligned} \tag{4.54}$$

where we have put

$$p = (\bar{p} - \pi p_o) \phi_1 \phi_2 \quad . \tag{4.55}$$

In the spirit of the developments outlined in the Appendix, we note in passing that $\bar{p}$ can be related to the pressure p^* in the three-dimensional theory. To see this, we write the constraint response for an incompressible fluid in the form

$$\underset{\sim}{\overline{T}}^{i} = -p^{*}g^{\frac{1}{2}}\underset{\sim}{g}^{i} \quad . \tag{4.56}$$

If we assume that $\underset{\sim}{\overline{n}}$ is the integrated resultant of $\underset{\sim}{\overline{T}}^{3}$ through the definition $(A13)_1$, then with the help of $(4.14)_1$ and $(4.26)_{1,2}$, we have

$$\overline{p} = \int_{\mathcal{a}} p^{*} d\theta^{1} d\theta^{2} = \frac{1}{\phi_1\phi_2}\int_{\mathcal{a}} p^{*} da \quad , \tag{4.57}$$

where da is the element of area in the normal cross-section of the jet. Thus, from a three-dimensional point of view, $\overline{p}\phi_1\phi_2$ is the resultant force due to the pressure p^{*} on a cross-section of the jet and $\overline{p}/\pi$ represents an average pressure. If we denote this average pressure by $\tilde{p}$, then (4.55) may be replaced by

$$p = \pi\phi_1\phi_2(\tilde{p}-p_o) \quad . \tag{4.58}$$

5. Additional kinematic considerations.

In this section, we examine in more detail the kinematics associated with the rotation of the directors in the plane normal to the axis of the jet. In particular, we appeal to the three-dimensional developments of the basic equations outlined in the Appendix to provide physical insight into the nature of the kinematic variables ω_1 and ω_2 defined by $(4.7)_3$ for the directed fluid jet.

Referred to the orthonormal basis $\underset{\sim}{e}_i$ introduced in section 2, the three-dimensional velocity field $\underset{\sim}{v}^*$ in (A2) can be expressed as

$$\underset{\sim}{v}^* = v^1\underset{\sim}{e}_1 + v^2\underset{\sim}{e}_2 + v^3\underset{\sim}{e}_3 \quad . \tag{5.1}$$

From (5.1) and (2.2), the component of the vorticity vector in the direction of the jet axis at time t is calculated to be

$$\tfrac{1}{2}(\text{curl } \underset{\sim}{v}^*) \cdot \underset{\sim}{e}_3 = \tfrac{1}{2}\left(\frac{\partial v^2}{\partial \bar{x}} - \frac{\partial v^1}{\partial \bar{y}}\right) = \tfrac{1}{2}\left[\frac{1}{\phi_1}\frac{\partial v^2}{\partial \theta^1} - \frac{1}{\phi_2}\frac{\partial v^1}{\partial \theta^2}\right] \quad . \tag{5.2}$$

Let $\underset{\sim}{D}$ denote the rate of deformation tensor[†] with components d_{ij} relative to $\underset{\sim}{e}_i$. Then, d_{12} represents the rate of shearing in the plane of the cross-section at time t and is given by

$$d_{12} = \underset{\sim}{e}_1 \cdot (\underset{\sim}{D} \cdot \underset{\sim}{e}_2) = \tfrac{1}{2}\left(\frac{\partial v^2}{\partial \bar{x}} + \frac{\partial v^1}{\partial \bar{y}}\right) = \tfrac{1}{2}\left[\frac{1}{\phi_1}\frac{\partial v^2}{\partial \theta^1} + \frac{1}{\phi_2}\frac{\partial v^1}{\partial \theta^2}\right] \quad . \tag{5.3}$$

Consider now the approximation $(A15)_1$ for the three-dimensional velocity field in the fluid. Using (4.4), (4.8) and (4.11), $(A15)_1$ takes the form

$$\underset{\sim}{v}^* = (w_1\theta^1 - \omega_2\phi_2\theta^2)\underset{\sim}{e}_1 + (w_2\theta^2 + \omega_1\phi_1\theta^1)\underset{\sim}{e}_2 + v\underset{\sim}{e}_3 \quad . \tag{5.4}$$

Substitution of the appropriate components of $\underset{\sim}{v}^*$ as given by (5.4) into (5.2) and (5.3) yields

---

[†]Recall that the rate of deformation tensor is defined as the symmetric part of the gradient of the velocity $\underset{\sim}{v}^*$.

$$\tfrac{1}{2}(\mathrm{curl}\ \underset{\sim}{v}^*) \cdot e_3 = \tfrac{1}{2}(\omega_1 + \omega_2) = \omega \quad ,$$
$$d_{12} = \tfrac{1}{2}(\omega_1 - \omega_2) = \gamma \quad . \tag{5.5}$$

Thus, the rotation components ω_1 and ω_2 of the director velocities may be combined in the manner indicated by $(5.5)_{1,2}$ to suggest physically meaningful quantities, and it may be useful to adopt ω and γ as alternative kinematic variables whenever interpretation is important. We call ω the _jet spin_ and γ the _sectional shearing_ in the jet.

The rotation of the directors may also give rise to a change in the orientation of the elliptical cross-section. In order to display this relationship explicitly, we need to dispose of some preliminary analysis based upon the results recorded in the Appendix. To this end, consider the position vector $\underset{\sim}{p}$ of a typical point in the three-dimensional body at some time τ. Recalling $(A14)_1$, as well as (4.1) and (4.2), it may be written as

$$\underset{\sim}{p}(\tau) = \underset{\sim}{p}(\theta^\alpha,\xi,\tau) = z\underset{\sim}{k} + (\theta^1\phi_1 \cos\psi_1 + \theta^2\phi_2 \cos\psi_2)\underset{\sim}{i} + (\theta^1\phi_1 \sin\psi_1 + \theta^2\phi_2 \sin\psi_2)\underset{\sim}{j} \ . \tag{5.6}$$

It follows that the position (x,y) occupied by the material particle (θ^1,θ^2) at time τ is given by

$$x = \theta^1\phi_1 \cos\psi_1 + \theta^2\phi_2 \cos\psi_2 \quad ,$$
$$y = \theta^1\phi_1 \sin\psi_1 + \theta^2\phi_2 \sin\psi_2 \quad , \tag{5.7}$$

where since we are concerned only with details in the cross-section of the jet we omit explicit reference to z or θ^3. Inverting (5.7), we obtain

$$
\begin{aligned}
\Delta\phi_1\,\theta^1 &= x\sin\psi_2 - y\cos\psi_2 \ , \\
\Delta\phi_2\,\theta^2 &= -x\sin\psi_1 + y\cos\psi_1 \ ,
\end{aligned}
\tag{5.8}
$$

where

$$
\Delta = \cos\psi_1\sin\psi_2 - \sin\psi_1\cos\psi_2 = \sin(\psi_1-\psi_2) \neq 0 \tag{5.9}
$$

and where use has been made of (4.3). Now the free surface is located by the time-invariant expression (4.28). Thus, after substituting (5.8) into (4.28), we can locate the position of the free surface in the plane of the normal cross-section of the jet at an arbitrary time τ. This leads to the expression

$$
\begin{aligned}
(\phi_2^2\sin^2\psi_2 + \phi_1^2\sin^2\psi_1)x^2 - (\phi_2^2\sin 2\psi_2 + \phi_1^2\sin 2\psi_1)xy \\
+ (\phi_2^2\cos^2\psi_2 + \phi_1^2\cos^2\psi_1)y^2 = (\Delta\,\phi_1\phi_2)^2 \ .
\end{aligned}
\tag{5.10}
$$

By a well-known formula,† the angle θ that a semiaxis of the ellipse (5.10) makes relative to the fixed Cartesian x-axis is given by

$$
\tan 2\theta = \frac{\phi_1^2\sin 2\psi_1 + \phi_2^2\sin 2\psi_2}{\phi_1^2\cos 2\psi_1 + \phi_2^2\cos 2\psi_2} \ . \tag{5.11}
$$

Thus, θ is a measure of the orientation of the elliptical cross-section of the jet and, in view of the discussion preceding (4.20), it is clear that θ in (5.11) may be identified with its counterpart in (4.9) at time t. We call θ the _sectional_ _orientation_ of the jet at a point ξ. Taking the material time derivative of (5.11), evaluating the result at time t and making use of (4.9), we obtain

$$
\dot{\theta} = \frac{\phi_2^2\omega_2 - \phi_1^2\omega_1}{\phi_2^2 - \phi_1^2} \tag{5.12}
$$

which is an expression for the time rate of change of the sectional orientation

---

†See, for example, Noble [9], p. 379.

at ξ. We call $\dot{\theta}$ the rate of sectional rotation. The twisting of the jet is associated with the spatial variation of θ. Hence, $\theta_z = \partial\theta/\partial z$ may be regarded as representing the local twist per unit length of the jet, or simply the twist.†

Before proceeding further, it is of interest to draw correspondence between an aspect of the kinematics of a directed curve as given by Ericksen and Truesdell [1] and the present developments for a twisted jet. In our notation, these authors have called

$$\mathbf{d}'_\alpha \cdot \mathbf{d}_\beta \tag{5.13}$$

the components of the wryness of the directors along the curve c of $\mathcal{R}$. With (4.10) and (5.13), for the twisted jet we have

$$\begin{aligned} \mathbf{d}'_1 \cdot \mathbf{d}_1 &= z'\phi_1\phi_{1z} \quad , \quad \mathbf{d}'_1 \cdot \mathbf{d}_2 = z'\phi_1\phi_2\theta_z \quad , \\ \mathbf{d}'_2 \cdot \mathbf{d}_1 &= - z'\phi_1\phi_2\theta_z \quad , \quad \mathbf{d}'_2 \cdot \mathbf{d}_2 = z'\phi_2\phi_{2z} \quad . \end{aligned} \tag{5.14}$$

Hence, in the special case when the length of each director is constant along c, the wryness corresponds to the twist.

Returning to (5.12), we note that this expression for $\dot{\theta}$ is a purely kinematical result and rests on the established connections between the theory of a directed curve and developments from the three-dimensional theory. With the help of (5.5) we may express (5.12) in the alternative and perhaps more revealing form

$$\dot{\theta} = \omega + \gamma\left(\frac{\phi_1^2 + \phi_2^2}{\phi_1^2 - \phi_2^2}\right) \quad . \tag{5.15}$$

This suggests that the sectional rotation is due to the sectional shearing as well as the jet spin.

---

†It may be noted that in general sectional rotation will cause a change in the twist. We also observe that even though $\dot{\theta}$ may vanish everywhere in the jet, the jet may still twist. This is because $\dot{\theta} = 0$ implies only that θ is a function of ξ and not that θ is constant.

Since the sectional orientation can be defined only when the jet is noncircular, (5.12) or (5.15) are valid as long as $\phi_1 \neq \phi_2$. The case of a circular jet must be treated separately. For this purpose, we first observe that the necessary and sufficient conditions for (5.10) to represent a circle are

$$\begin{aligned} \phi_2^2 \sin 2\psi_2 + \phi_1^2 \sin 2\psi_1 &= 0 \quad , \\ \phi_2^2 \cos 2\psi_2 + \phi_1^2 \cos 2\psi_1 &= 0 \quad . \end{aligned} \tag{5.16}$$

Since ϕ_1 and ϕ_2 cannot vanish in view of $(3.1)_3$ and (4.10), it follows that

$$\tan 2\psi_1 = \tan 2\psi_2 \tag{5.17}$$

and the only solutions of (5.17) consistent with (4.3) are

$$\psi_1 = \psi_2 \pm \frac{\pi}{2} \quad . \tag{5.18}$$

Now if we substitute (5.18) into (5.16), we obtain the result

$$\phi_1 = \phi_2 = \phi \quad , \tag{5.19}$$

where ϕ is the radius of the circular jet. Using (5.18) and $(4.6)_3$ in (5.5), we also have

$$\omega_1 = \omega_2 = \omega \quad , \quad \gamma = 0 \quad . \tag{5.20}$$

The results (5.16) to (5.20) are valid for any time τ. In particular, recalling (4.9) and (5.18), we may identify θ with ψ_1 and write for all time

$$\dot{\theta} = \dot{\psi}_1 = \dot{\psi}_2 = \omega \quad , \tag{5.21}$$

where θ refers only to the orientation of the orthogonal director pair since the cross-section is now completely symmetric.

Therefore, in the special case of a circular jet we must replace (5.5) and (5.12) by (5.20) and (5.21), respectively. The result (5.21) cannot be obtained directly from (5.12) or (5.15) since each has a singularity at $\phi_1 = \phi_2$. In a circular jet, $\dot{\theta}$ is associated with the rate of rotation of the director pair which, by (5.21), is the same as the jet spin; and a circular jet twists if the orientation of the director pair is not constant along its length. Of course, due to the symmetry of a circular jet, evidence of twist will not be apparent in the shape of the free surface.

In certain problems, the jet may experience a transition from an elliptical cross-section to a circular one. This may be accommodated by requiring θ to be continuous at the transition.

6. An inviscid directed fluid jet: Some general results.

The stress response in the three-dimensional theory of an inviscid fluid involves only a hydrostatic pressure. If the fluid is also incompressible, then the pressure becomes an arbitrary constraint response function and the determinate part of the stress response vanishes identically. Keeping this in mind, as in [3], we define an inviscid incompressible directed fluid jet by the assumption that

$$\hat{\underset{\sim}{n}} = \underset{\sim}{0} \quad , \quad \hat{\underset{\sim}{\pi}}^{\alpha} = \underset{\sim}{0} \quad , \quad \hat{\underset{\sim}{p}}^{\alpha} = \underset{\sim}{0} \quad . \tag{6.1}$$

Substituting these values into (4.52) to (4.54), we obtain

$$\begin{aligned}
-p_z - \pi\rho^* \phi_1\phi_2 g &= \phi_2\phi_{1z} h(\phi_1,\phi_2,\theta_z) + \phi_1\phi_{2z} h(\phi_2,\phi_1,\theta_z) \\
&\quad + \theta_z(\phi_2^2 - \phi_1^2) m(\phi_1,\phi_2,\theta_z) + \pi\rho^*\phi_1\phi_2\dot{v} \quad , \\
p + \phi_1\phi_2 h(\phi_1,\phi_2,\theta_z) &= \tfrac{1}{4}\pi\rho^*\phi_1^3\phi_2(\dot{\zeta}_1 + \zeta_1^2 - \omega_1^2) \quad , \\
-m(\phi_1,\phi_2,\theta_z) &= \tfrac{1}{4}\pi\rho^*\phi_1\phi_2(\dot{\omega}_1 + 2\zeta_1\omega_1) \quad , \\
p + \phi_1\phi_2 h(\phi_2,\phi_1,\theta_z) &= \tfrac{1}{4}\pi\rho^*\phi_2^3\phi_1(\dot{\zeta}_2 + \zeta_2^2 - \omega_2^2) \quad , \\
-m(\phi_2,\phi_1,\theta_z) &= \tfrac{1}{4}\pi\rho^*\phi_1\phi_2(\dot{\omega}_2 + 2\zeta_2\omega_2) \quad ,
\end{aligned} \tag{6.2}$$

where we have omitted the superposed star from the gravitational constant g. In a complete theory, the equations of motion (6.2) must be supplemented by the incompressibility condition (4.16) and the results (4.8) and (5.12). In the rest of this paper, for convenience we refer to the straight, inviscid incompressible directed fluid jet characterized by (6.2), (4.16), (4.8) and (5.12) as an <u>ideal jet</u>.

We have already seen from (5.20) that the sectional shearing γ vanishes identically in the special case of a circular jet. With reference to an ideal jet, we now ask if there are other circumstances for which the sectional shearing is also zero. For simplicity, we omit surface tension and set

$$h = m = 0 \quad . \tag{6.3}$$

When $\gamma = 0$, it follows from $(5.5)_2$ that

$$\omega_1 = \omega_2 = \omega \quad . \tag{6.4}$$

Then, provided ω does not vanish, from $(6.2)_{3,5}$ we find

$$\zeta_1 = \zeta_2 = \zeta \quad , \tag{6.5}$$

where ζ is introduced for convenience. With the help of (6.3) to (6.5), equations $(6.2)_2$ and $(6.2)_4$ may be combined to yield

$$(\phi_1^2 - \phi_2^2)(\dot{\zeta} + \zeta^2 - \omega^2) = 0 \quad . \tag{6.6}$$

Hence, we conclude that

$$\phi_1 = \phi_2 \quad , \tag{6.7}$$

unless

$$\dot{\zeta} + \zeta^2 - \omega^2 = 0 \quad . \tag{6.8}$$

When (6.8) is not satisfied, it follows from (6.4) and (6.7) that the jet is circular. It remains then to examine the implications of (6.8). From (6.8), (6.3) to (6.5) and $(6.2)_2$, we have $p = 0$ and integration of $(6.2)_1$ gives

$$v = gt + f_1(\xi) \quad , \tag{6.9}$$

where f_1 is an arbitrary function of ξ. Next, from $(4.7)_1$ we have

$$z = \frac{1}{2} gt^2 + f_1(\xi)t + f_2(\xi) \quad ,$$
$$z' = f_1'(\xi)t + f_2'(\xi) \quad , \tag{6.10}$$

where f_2 is an arbitrary function of ξ and is such that† $f_2'(\xi) \neq 0$ for all ξ. Using (6.5) and (6.9) in (4.16), we obtain

$$\zeta = - f_1'/2z' \tag{6.11}$$

and with (6.8) and (6.10) this gives

$$\omega^2 = 3\zeta^2 \quad . \tag{6.12}$$

Then, substitution of (6.3), (6.4) and (6.10) to (6.12) into $(6.2)_3$ yields

$$\omega = \zeta = 0 \quad , \quad f_1(\xi) = \text{const.} \tag{6.13}$$

Thus, if (6.7) does not hold, then ω and γ are both zero and by (5.12) we also have

$$\dot{\theta} = 0 \quad . \tag{6.14}$$

The above results may be summarized as

Theorem 6.1. *If the sectional shearing γ vanishes at a point ξ in an ideal jet without surface tension, then either the jet is circular at ξ or both the jet spin ω and the rate of sectional rotation $\dot{\theta}$ are zero at ξ.*

As a consequence of the above theorem, we observe that when the jet spin is nonzero at a noncircular section of an ideal jet without surface tension, the sectional shearing cannot vanish there. Hence follows

Corollary 6.1. *An ideal jet without surface tension can rotate rigidly about its axis only if it is circular.*

The last result also follows from the three-dimensional theory. To see this,

†This restriction on f_2 is due to $(3.1)_3$.

we observe that when a three-dimensional body rotates rigidly, the velocity field in the body is completely symmetric with respect to the axis of rotation. Hence the three-dimensional pressure of a rigidly rotating jet will have rotational symmetry about the jet axis. To satisfy the boundary condition that the pressure be constant at the free surface, the jet must be circular.

In order to proceed further, we need two integrals of the governing differential equations (6.2) which will be obtained in the absence of surface tension. Thus, with $m = 0$ and the use of (4.8), $(6.2)_{3,5}$ reduce to

$$\dot{\omega}_1 + 2(\dot{\phi}_1/\phi_1)\omega_1 = 0 \quad , \quad \dot{\omega}_2 + 2(\dot{\phi}_2/\phi_2)\omega_2 = 0 \quad . \tag{6.15}$$

These equations may be integrated in the form

$$\phi_1^2\omega_1 = c_1(\xi) \quad , \quad \phi_2^2\omega_2 = c_2(\xi) \quad , \tag{6.16}$$

where c_1 and c_2 are arbitrary functions of ξ. An immediate consequence of $(6.16)_{1,2}$ and (5.5) is the following

Theorem 6.2. *If at some time t^* both the jet spin ω and sectional shearing γ vanish at a point ξ of an ideal jet without surface tension, then they must remain zero at ξ for all time.*

This result is similar to one expressing permanence of irrotational motion in three-dimensional inviscid fluid theory (see, e.g., Milne-Thomson [8 , p. 86]). We note, however, that in addition to ω, the sectional shearing γ must vanish at ξ to ensure that the jet spin at ξ remain zero for all time.

In the special case of a circular jet, (4.47) implies that the function m vanishes even in the presence of surface tension. Hence, corresponding to (6.16), in this case we have

$$\phi^2\omega = c(\xi) \quad , \tag{6.17}$$

where ϕ is the radius of the circular jet given by (5.19) and c is an arbitrary function of ξ. From (6.17) follows

Theorem 6.3. <u>If at some time t^* the jet spin ω vanishes at a point ξ of an ideal circular jet with surface tension, then it must remain zero at ξ for all time. Moreover, if the jet spin at ξ is not zero at some time t^*, then ω cannot vanish at ξ for all time.</u>

This last result is much stronger than Theorem 6.2, but it is limited to the special case of a circular jet.

Prior to a statement of the next theorem, we need to dispose of some preliminary results. With reference to a particular material point ξ, we examine first the temporal continuity of the various field quantities associated with the directed fluid jet. To do this, we allow a point of discontinuity s to move with velocity u along the jet. Provided that the point of discontinuity is not material (that is, $u \neq v$), the field quantities of the jet may be discontinuous at a certain instant of time, say $\hat{t}$, when s is coincident with ξ.

Jump conditions for a general directed curve can be derived by the usual procedure. For our present purpose, we record them here in a form appropriate for the case in which the curve c is fixed in space, i.e.,

$$\begin{aligned} [\![\rho(v-u)]\!] &= 0 \quad , \\ [\![\rho \underset{\sim}{v}(v-u) - \underset{\sim}{n}]\!] &= \underset{\sim}{0} \quad , \\ [\![\rho y^{\alpha\beta} \underset{\sim}{w}_{\beta}(v-u) - \underset{\sim}{p}^{\alpha}]\!] &= \underset{\sim}{0} \quad , \end{aligned} \tag{6.18}$$

where v is the component of the velocity vector tangent to c and where we have introduced the notation

$$[\![\psi]\!] = \psi(\xi, \hat{t}^{+}) - \psi(\xi, \hat{t}^{-}) \quad . \tag{6.19}$$

In the case of the straight, inviscid incompressible fluid jet under consideration, substitution of $(3.14)_3$, (4.4), (4.8), (4.11), (4.14), (4.21), (4.49), (4.51) and (6.1) into (6.18) yields

$$
\begin{aligned}
&[\![\rho^* \pi \phi_1 \phi_2 (v-u)]\!] = 0 \quad , \\
&[\![\rho^* \pi \phi_1 \phi_2 v(v-u) + \bar{p} \phi_1 \phi_2]\!] = 0 \quad , \\
&[\![\tfrac{1}{4}\, \rho^* \pi \phi_1 \phi_2 \dot{\phi}_\alpha (v-u)]\!] = 0 \quad , \quad (\alpha = 1,2) \quad , \\
&[\![\tfrac{1}{4}\, \rho^* \pi \phi_1 \phi_2 \phi_\alpha \omega_\alpha (v-u)]\!] = 0 \quad , \quad (\alpha = 1,2) \quad .
\end{aligned}
\tag{6.20}
$$

We now restrict attention to a discontinuity in which

$$[\![\phi_\alpha]\!] = 0 \quad (\alpha = 1,2) \quad , \tag{6.21}$$

i.e., a situation in which the lengths of the semiaxes of the elliptical cross-section are continuous at $\hat{t}$. This is a mild restriction from the point of view of a free jet. In view of (6.21), from $(6.20)_1$ we obtain

$$[\![v]\!] = 0 \tag{6.22}$$

and this, together with $(6.20)_2$, (6.21) and (4.55), results in

$$[\![\bar{p}]\!] = [\![p]\!] = 0 \quad . \tag{6.23}$$

Also, the last two conditions of (6.20) yield

$$[\![\dot{\phi}_\alpha]\!] = 0 \quad , \quad [\![\omega_\alpha]\!] = 0 \quad , \quad (\alpha = 1,2) \quad . \tag{6.24}$$

Hence, the assumption of continuity for ϕ_α implies that each of the variables $v, \bar{p}, p, \dot{\phi}_\alpha$ and ω_α are continuous functions of time. By appealing to the equations of motion, we can arrive at a further conclusion regarding the components of the director accelerations $\ddot{\phi}_\alpha$. In the absence of surface tension, $(6.2)_{2,4}$ reduce to

$$p = \frac{1}{4}\pi\rho^* \phi_1^2\phi_2(\ddot{\phi}_1 - \phi_1\omega_1^2) \quad ,$$
$$p = \frac{1}{4}\pi\rho^* \phi_2^2\phi_1(\ddot{\phi}_2 - \phi_2\omega_2^2) \quad , \tag{6.25}$$

where (4.8) has been used to eliminate ζ_α. Then, combining (6.21) and (6.23) to (6.25), we obtain

$$[[\ddot{\phi}_\alpha]] = 0 \tag{6.26}$$

so that $\ddot{\phi}_\alpha$ is also continuous at $\hat{t}$.

Consider now a particular material point ξ of the jet and suppose that for some time interval

$$t_1 < t \leqq t_o \tag{6.27}$$

the jet is circular at ξ. Then (5.18) and (5.19) must hold for all t satisfying (6.27) and by differentiating each of these relations with respect to time, we obtain

$$\dot{\phi}_1 = \dot{\phi}_2 \quad , \quad \omega_1 = \omega_2 \quad , \quad t_1 < t \leqq t_o \quad . \tag{6.28}$$

In particular, we have

$$\phi_1 = \phi_2 \quad , \quad \dot{\phi}_1 = \dot{\phi}_2 \quad , \quad \omega_1 = \omega_2 \quad , \quad c_1 = c_2 \quad \text{at} \quad t = t_o \quad , \tag{6.29}$$

where $(6.29)_4$ follows from (6.16). With the help of $(6.29)_4$, the expressions $(6.25)_{1,2}$ and $(6.16)_{1,2}$ can be combined in the single differential equation

$$\phi_1\ddot{\phi}_1 - \phi_2\ddot{\phi}_2 + c^2\left(\frac{1}{\phi_2^2} - \frac{1}{\phi_1^2}\right) = 0 \quad , \tag{6.30}$$

where we have set $c = c_1 = c_2$.

In what follows, we hold ξ fixed at all times and treat (6.30) as an ordinary differential equation subject to the initial conditions (6.29). Introducing the change of variables

$$\delta = \tfrac{1}{2}(\phi_1 - \phi_2) \quad , \quad \phi = \tfrac{1}{2}(\phi_1 + \phi_2) \quad , \tag{6.31}$$

(6.30) can be rewritten in the form

$$\phi\ddot{\delta} + \delta\{\ddot{\phi} + \frac{2c^2\phi}{(\phi^2 - \delta^2)^2}\} = 0 \tag{6.32}$$

and, in terms of the variable δ, the initial conditions (6.29) become

$$\delta(t_o) = 0 \quad , \quad \dot{\delta}(t_o) = 0 \quad . \tag{6.33}$$

Next, we express (6.32) in the form

$$\ddot{\delta} = f(\ddot{\phi}, \phi, \delta) \quad , \tag{6.34}$$

where

$$f = -\delta\{\frac{\ddot{\phi}}{\phi} + 2[\frac{c}{\phi^2 - \delta^2}]^2\} \quad . \tag{6.35}$$

Now, the function f depends upon δ and also implicitly upon time through the arguments $\ddot{\phi}$ and ϕ. Clearly f is a continuous function of δ and by virtue of (6.21) and (6.26) it also depends continuously on t. Moreover, the partial derivative of f with respect to δ is always continuous and bounded.† The above conditions are sufficient to ensure that f satisfies a Lipschitz condition. Hence, by a uniqueness theorem of ordinary differential equations (see, for example, Corollary 1, p. 84 of Rosenlicht [10]), there is at most one solution to (6.32) that also satisfies (6.33). By inspection, this solution is the trivial one given by

$$\delta(t) = 0 \quad , \quad (-\infty < t < \infty) \quad . \tag{6.36}$$

†The singular points $\phi = \pm\,\delta$ may be ruled out since they imply $\phi_2 = 0$ and $\phi_1 = 0$, respectively.

This means that if the conditions (6.29) hold for a given material point ξ at some time t_0, then the jet will remain circular at that point for all time. We can make a further observation, however, regarding points at which the jet may be noncircular at some time during its motion. Suppose that the jet is noncircular at ξ for some time t^*. Then, it follows from the above analysis that the jet can be circular at ξ at another instant of time, say $\bar{t}$, only if at $\bar{t}$ all the conditions (6.29) are not satisfied simultaneously, i.e., if either

$$\dot{\phi}_1 \neq \dot{\phi}_2 \quad \text{or} \quad \omega_1 \neq \omega_2 \quad \text{when} \quad \phi_1 = \phi_2 \ . \tag{6.37}$$

With the aid of (6.25), the condition (6.37) may be restated as

$$\dot{\phi}_1 \neq \dot{\phi}_2 \quad \text{or} \quad \ddot{\phi}_1 \neq \ddot{\phi}_2 \quad \text{when} \quad \phi_1 = \phi_2 \ . \tag{6.38}$$

It is clear from (6.38) that if a noncircular section should later become circular, it can remain circular for only an isolated instant of time. In summary, at each material point, the jet is either circular for all time or it is always noncircular, except possibly at isolated instants of time. Therefore, for an ideal jet and in the absence of surface tension, we can unambiguously classify each material point as belonging either to a <u>circular</u> or to a <u>noncircular</u> jet, depending upon the initial conditions assigned to the jet.

With the above preliminary background, we can now prove

Theorem 6.4. <u>Let an ideal jet without surface tension be noncircular at some material point ξ and time t^* and let the rate of sectional rotation $\dot{\theta}(\xi,t^*)=0$. Then, the sectional orientation θ at ξ will have the same constant value for all time.</u>

Proof. From the remarks preceding the statement of the theorem, we know that the jet will always be noncircular at ξ except possibly at isolated instants of time. If we exclude such times for the moment, then $\dot{\theta}$ is always given by (5.18), which may be written in the form

$$\dot{\theta}(\phi_1^2 - \phi_2^2) = c_1 - c_2 \quad , \tag{6.39}$$

where in obtaining (6.39) we have also used $(6.16)_{1,2}$. At time t^*, $\dot{\theta} = 0$ at ξ and hence the right-hand side of (6.39) vanishes identically. It then follows that at the material point ξ,

$$\theta = \text{const.} \quad \text{whenever} \quad \phi_1 \neq \phi_2 \quad . \tag{6.40}$$

But we can only have $\phi_1 = \phi_2$ at ξ at an isolated instant of time, so that θ must be continuous at these times. Therefore θ must have the same constant value for all time and the theorem is proved.

An important consequence of Theorem 6.4 can be stated as

Corollary 6.2. *At some time t^* let an ideal jet without surface tension be non-circular everywhere except possibly at isolated points along its length. If the sectional rotation vanishes everywhere at time t^* and if the jet does not twist at time t^*, then the jet must be without twist at all times during its motion.*

7. <u>Solutions in some special cases.</u>

We discuss here some special solutions for circular and elliptical jets. We begin with the former and recall from (5.19), (5.20) and (4.8) that in the case of a circular jet

$$\phi_1 = \phi_2 = \phi \quad , \quad \zeta_1 = \zeta_2 = \zeta \quad , \quad \omega_1 = \omega_2 = \omega \quad . \tag{7.1}$$

After substituting $(7.1)_1$ into (2.6), the expression for q reduces to

$$q = T\left[\frac{\phi_{zz}}{(1+\phi_z^2)^{3/2}} - \frac{1}{\phi(1+\phi_z^2)^{1/2}}\right] \tag{7.2}$$

and the definitions (4.44) and (4.45) become

$$h = \pi q \quad , \quad m = 0 \quad . \tag{7.3}$$

With the use of (7.1) and (7.3), the system of differential equations (6.2), (4.16), (4.8) and (5.21) in the case of a circular jet reduces to

$$\begin{gathered} -p_z - \pi\rho^* g\phi^2 = \pi q(\phi^2)_z + \pi\rho^*\phi^2\dot{v} \quad , \quad p + \pi q\phi^2 = \tfrac{1}{4}\pi\rho^*\phi^4(\dot{\zeta} + \zeta^2 - \omega^2) \quad , \\ \dot{\omega} + 2\zeta\omega = 0 \quad , \quad 2\zeta + v_z = 0 \quad , \quad \dot{\phi} = \zeta\phi \quad , \quad \dot{\theta} = \omega \quad . \end{gathered} \tag{7.4}$$

A simple solution of the system of equations (7.4) in the absence of gravity is

$$\begin{gathered} v = v_o \quad , \quad \phi = a \quad , \quad \zeta = 0 \quad , \quad \omega = \omega_o \quad , \\ p = \pi a^2(\tilde{p} - p_o) = -\tfrac{1}{4}\pi\rho^* a^4\omega_o^2 + \pi a T \quad , \end{gathered} \tag{7.5}$$

where use has been made of (4.58) and where v_o, a and ω_o are all constants. With the help of $(7.5)_5$ and $(7.4)_6$, the rate of rotation of the director pair can be expressed in terms of the average pressure $\tilde{p}$ in the form

$$\dot{\theta} = \theta_t + v_o\theta_z = \omega_o = \pm 2\left[\frac{(p_o - \tilde{p}) + T/a}{\rho^* a^2}\right]^{\frac{1}{2}} \quad . \tag{7.6}$$

An expression for the orientation θ of the director pair is obtained by integrating (7.6) in the form

$$\theta = \Omega_1 t + (\Omega_2/v_o)z \tag{7.7}$$

where

$$\Omega_1 + \Omega_2 = \omega_o \quad . \tag{7.8}$$

It is evident from (7.7) that the rotation of the directors can be decomposed into a rigid rotation Ω_1 about the axis of the jet plus a steady motion represented by Ω_2, where the directors twist linearly along its length. In the absence of surface tension, (7.5) is the same solution as that given by Green and Laws [5] where a further discussion of these relations may be found. We remark, however, that when $v_o = 0$, only the rigid part of this motion will persist. Then due to the rotational symmetry of the director pair in the plane of the cross-section, we may put $\theta_z = 0$ without loss in generality. Since v_o can be made to vanish with a suitable choice of reference frame, it follows that the motion (7.7) is dynamically equivalent to rigid rotation alone, i.e.,

$$\theta = \Omega_1 t \quad . \tag{7.9}$$

Next, we examine steady solutions of (7.4) and suppress explicit dependence upon time in all functions. For such motions, the system of equations (7.4) can be rewritten in the form

$$\begin{gathered} -p_z - \pi\rho^* g\phi^2 = \pi q(\phi^2)_z + \pi\rho^*\phi^2 v v_z \quad , \quad p + \pi\phi^2 q = \tfrac{1}{4}\pi\rho^*\phi^4(v\zeta_z + \zeta^2 - \omega^2) \quad , \\ v\omega_z + 2\zeta\omega = 0 \quad , \quad 2\zeta + v_z = 0 \quad , \quad v\phi_z = \phi\zeta \quad , \quad v\theta_z = \omega \quad . \end{gathered} \tag{7.10}$$

From the combination of $(7.10)_4$ and $(7.10)_5$, we have the immediate integral

$$\phi^2 v = k \quad , \tag{7.11}$$

where k is a constant. It is clear from (7.11) that the velocity vanishes only if the jet is everywhere at rest. In conjunction with $(7.10)_6$, this implies that the director pair rotates if and only if it twists.

Elimination of ζ between $(7.10)_{3,4}$ gives

$$v\omega_z - v_z\omega = 0 \quad , \tag{7.12}$$

and this can be integrated to yield

$$\omega = \alpha v \quad , \tag{7.13}$$

where α is a constant. From (7.13) and $(7.10)_6$ it follows that

$$\theta_z = \alpha \quad , \tag{7.14}$$

as long as the jet is not at rest. We conclude, therefore, that in steady motion of an ideal circular jet the twist per unit length is constant.

Returning to (7.13) and using (7.11), we can express ω in terms of ϕ by

$$\omega = k\alpha/\phi^2 \quad . \tag{7.15}$$

Substituting (7.15) into $(7.10)_2$, we obtain

$$[p + \tfrac{1}{4}\pi\rho^*\alpha^2k^2] + \pi\phi^2 q = \tfrac{1}{4}\pi\rho^*\phi^4(v\zeta_z + \zeta^2) \quad . \tag{7.16}$$

Now put

$$\hat{p} = p + \tfrac{1}{4}\pi\rho^*\alpha^2k^2 \tag{7.17}$$

so that $\hat{p}_z = p_z$. Then, the system of equations (7.10) may be written as

$$\begin{gathered} \hat{p}_z - \pi\rho^* g\phi^2 = \pi q(\phi^2)_z + \pi\rho^*\phi^2 vv_z \quad , \quad \hat{p} + \pi\phi^2 q = \tfrac{1}{4}\pi\rho^*\phi^4(v\zeta_z + \zeta^2) \quad , \\ v_z + 2\zeta = 0 \quad , \quad v\phi_z = \phi\zeta \quad . \end{gathered} \tag{7.18}$$

Apart from the difference in the pressure terms, the above equations are formally equivalent to those appropriate for steady motion of an ideal circular jet in the absence of twist. Hence we have the following theorem of correspondence:

Theorem 7.1. *Any steady solution for a nontwisting ideal circular jet will also be a solution for a twisting jet with the twist given by (7.14), provided that p is replaced by $\hat{p}$ defined by (7.17). Conversely, any steady solution of (7.4) with a nonvanishing twist must have a constant twist per unit length and have ϕ and v given by the steady solution (7.4) with $\omega = 0$.*

This result has important implications in certain instances. For example, consider steady flow from a circular hole in the bottom of a large tank. Due to the Coriolis effect, a slight twist will be imparted to the jet as it leaves the tank. But, according to Theorem 7.1 we may ignore the effect of the twist if only information regarding the jet radius or velocity distribution is desired.

We now turn to the general elliptical jet governed by the system of equations (6.2), (4.16), (4.8) and (5.12). If we neglect gravity as well as surface tension, these equations have the simple solution

$$\begin{gathered} \omega_1 = \hat{\omega}_1 \quad , \quad \omega_2 = \hat{\omega}_2 \quad , \\ \phi_1 = a \quad , \quad \phi_2 = b \quad , \quad \zeta_1 = \zeta_2 = 0 \quad , \quad v = v_o \quad , \\ -p = \tfrac{1}{4}\pi\rho^* ab(a^2\hat{\omega}_1^2) = \tfrac{1}{4}\pi\rho^* ab(b^2\hat{\omega}_2^2) \quad , \end{gathered} \tag{7.19}$$

where $a, b, \hat{\omega}_1, \hat{\omega}_2$ and v_o are all constants and must satisfy the condition

$$a\hat{\omega}_1 = \pm\, b\hat{\omega}_2 \quad . \tag{7.20}$$

It is perhaps more revealing to express (7.20) in terms of the alternative kinematic variables ω and γ introduced in section 5. From (5.5), (7.19) and (7.20), we see that in the present solution both ω and γ are constants and related by

$$\gamma = \left[\frac{b-a}{b+a}\right]^{\pm 1} \omega \quad , \tag{7.21}$$

where the $\pm$ sign corresponds to that in (7.20). In order to get an understanding of the physical meaning of (7.21), we examine the particular case in which $|a-b|$ is small, or the jet is nearly circular. Then, corresponding to the plus sign in (7.21), the ratio γ/ω is small, indicating a preponderance of jet spin accompanied by a relatively small sectional shearing. The minus sign, on the other hand, corresponds to more sectional shearing than jet spin.

With reference to the plus sign in (7.20), the sectional rotation (5.12) described by the solution (7.19) is given by

$$\dot{\theta} = \Omega_{(+)} = \frac{b^2\hat{\omega}_2 - a^2\hat{\omega}_1}{b^2 - a^2} = \frac{\hat{\omega}_1 a}{a+b} \quad . \tag{7.22}$$

With the help of $(7.19)_7$ and (4.58), this can be written as

$$\dot{\theta}^2 = \Omega^2_{(+)} = -\frac{4p}{\pi\rho^* ab(a+b)^2} = \frac{4(p_o - \tilde{p})}{\rho^*(a+b)^2} \quad . \tag{7.23}$$

Thus, as might be expected, the average pressure $\tilde{p}$ must be less than p_o to sustain the motion (7.19). Again referring only to the case when $a\hat{\omega}_1 = b\hat{\omega}_2$, we may express ω and γ in terms of $p_o - \tilde{p}$ in the form

$$\omega^2 = \omega^2_{(+)} = \frac{1}{\rho^*}(p_o - \tilde{p})(\frac{1}{a} + \frac{1}{b})^2 \quad , \quad \gamma^2 = \gamma^2_{(+)} = \frac{1}{\rho^*}(p_o - \tilde{p})(\frac{1}{a} - \frac{1}{b})^2 \quad . \tag{7.24}$$

When $a\hat{\omega}_1 = -b\hat{\omega}_2$, the relations corresponding to (7.23) and (7.24) are

$$\dot{\theta}^2 = \Omega^2_{(-)} = \frac{4(p_o - \tilde{p})}{\rho^*(a-b)^2} \tag{7.25}$$

and

$$\omega^2 = \omega^2_{(-)} = \frac{1}{\rho^*}(p_o - \tilde{p})(\frac{1}{a} - \frac{1}{b})^2 \quad , \quad \gamma^2 = \gamma^2_{(-)} = \frac{1}{\rho^*}(p_o - \tilde{p})(\frac{1}{a} + \frac{1}{b})^2 \quad . \tag{7.26}$$

Hence, the solution separates into two distinct cases which, using an obvious notation, we have distinguished by the subscripts (+) and (-). Proceeding as we did after (7.6), it is possible to show that when the motion is steady, both (7.23) and (7.25) yield a constant twist. From (7.23) and (7.25) it is also clear that in each case a twist of either sense is possible. Two unsteady motions (corresponding to $\theta = \Omega_{(\pm)}t$) in which the noncircular jet appears to rotate as a rigid body about its axis are also possible.[†] Of course, any linear combination of both the steady and unsteady motions can occur. The steady motion is illustrated in Fig. 1. Because the jet is noncircular, the configuration of its free surface in the present solution appears more striking than in the corresponding solution (7.5) for a circular jet.

We close with a few remarks concerning the behavior of the two solutions (7.24) and (7.26) in the limit $|a-b| \to 0$. When $a = b$, the jet is circular with a constant radius. From $(5.20)_1$, this requires that $\omega_1 = \omega_2$, corresponding to the plus sign in (7.20) and the first of the two solutions above. One would expect, then, that we cannot properly take (7.25) and (7.26) to this limit. The fact that (7.25) is singular at $a = b$ seems to bear this out. On the other hand, taking the limit of (7.23) and (7.24) as $b \to a$, we find

$$\Omega^2_{(+)} \to \frac{(p_o - \tilde{p})}{\rho^* a^2} \quad , \quad \omega^2_{(+)} \to \frac{4(p_o - \tilde{p})}{\rho^* a^2} \quad , \quad \gamma^2_{(+)} \to 0 \quad . \tag{7.27}$$

In view of (5.21), the second of (7.27) is consistent with (7.6) (when $T = 0$) and the third agrees with $(5.20)_2$. The first of (7.27) really has no significance since there is no discernible orientation of the cross-section when $a = b$.

Acknowledgement. The results reported here were obtained in the course of research supported by the U.S. Office of Naval Research under Contract N00014-76-C-0474, Project NR 062-534, with the University of California, Berkeley.

[†] By Corollary 6.1, a truly rigid motion is possible only when $a = b$. However, since θ measures the local orientation of the ellipse that forms the free surface of the jet, a solution of the form $\theta = \Omega_{(\pm)}t$ would give the appearance of a rigid body rotation about the jet axis.

References

1. Ericksen, J. L., and Truesdell, C., Exact theory of stress and strain in rods and shells. Arch. Rational Mech. Anal. 1, 295-323 (1958).

2. Green, A. E., Compressible fluid jets, Arch. Rational Mech. Anal. 59, 189-205 (1975).

3. Green, A. E., On the non-linear behavior of fluid jets, Int. J. Engng. Sci. 14, 49-63 (1976).

4. Green, A. E., On the steady motion of jets with elliptical sections, Acta Mechanica, to appear.

5. Green, A. E., and Laws, N., Ideal fluid jets, Int. J. Engng. Sci. 6, 317-328 (1968).

6. Green, A. E., Naghdi, P. M., and Wenner, M. L., On the theory of rods, I. Derivations from the three-dimensional equations, Proc. R. Soc. Lond. A 337, 451-483 (1974).

7. Green, A. E., Naghdi, P. M., and Wenner, M. L., On the theory of rods, II. Developments by direct approach, Proc. R. Soc. Lond. A 337, 485-507 (1974).

8. Milne-Thomson, L. M., Theoretical hydrodynamics, 5th ed., MacMillan, 1968.

9. Noble, B., Applied linear algebra, Prentice-Hall, 1969.

10. Rosenlicht, M., Introduction to analysis, Scott, Foresman and Co., 1968.

11. Schlichting, H., Boundary layer theory, 6th ed., McGraw-Hill, 1968.

12. Taylor, G. I., The boundary layer in the converging nozzle of a swirl atomizer, Quart. J. Mech. Appl. Math. 3, 129-139 (1950).

Appendix

In this appendix we record certain details of the approximation procedure [6] whereby equations governing the motion of a rod-like body can be obtained by integration from the three-dimensional equations.

Let the material particles of a three-dimensional body be identified with the convected coordinates θ^i $(i = 1,2,3)$ and let $\underset{\sim}{p}$ denote the position vector to a typical point in the body relative to a fixed origin at time t. Then

$$\underset{\sim}{p} = \underset{\sim}{p}(\theta^1, \theta^2, \theta^3, t) \quad , \quad \underset{\sim}{g}_i = \frac{\partial \underset{\sim}{p}}{\partial \theta^i} \quad ,$$
$$g_{ij} = \underset{\sim}{g}_i \cdot \underset{\sim}{g}_j \quad , \quad \underset{\sim}{g}^i \cdot \underset{\sim}{g}_j = \delta^i_j \quad , \quad g^{ij} = \underset{\sim}{g}^i \cdot \underset{\sim}{g}^j \quad , \quad g = \det g_{ij} \quad , \tag{A1}$$

where $\underset{\sim}{g}_i$ and $\underset{\sim}{g}^j$ are respectively the covariant and contravariant base vectors, g_{ij} is the metric tensor, g^{ij} is its inverse and δ^i_j is the Kronecker delta in 3-space. The velocity vector is defined by

$$\overset{*}{\underset{\sim}{v}} = \dot{\underset{\sim}{p}} \quad , \tag{A2}$$

where a superposed dot designates the material time derivative holding θ^i fixed. The local field equations of the three-dimensional theory are

$$\overline{\overset{*}{\rho} g^{\frac{1}{2}}}^{\,\boldsymbol{\cdot}} = 0 \quad ,$$
$$\underset{\sim}{T}^i{}_{,i} + \overset{*}{\rho} \underset{\sim}{f} g^{\frac{1}{2}} = \overset{*}{\rho} \dot{\overset{*}{\underset{\sim}{v}}} g^{\frac{1}{2}} \quad , \tag{A3}$$
$$\underset{\sim}{g}_i \times \underset{\sim}{T}^i = \underset{\sim}{0} \quad ,$$

where $\overset{*}{\rho}$ is the three-dimensional mass density, $\overset{*}{\underset{\sim}{f}}$ is the body force vector and a comma denotes partial differentiation with respect to θ^i. Also, the stress vector $\underset{\sim}{t}$ on a surface whose outward unit normal is $\overset{*}{\underset{\sim}{\nu}}$ is given by

$$\underset{\sim}{t} = \underset{\sim}{T}^i \overset{*}{\nu}_i g^{-\frac{1}{2}} \quad , \quad \overset{*}{\underset{\sim}{\nu}} = \overset{*}{\nu}_i \underset{\sim}{g}^i = \overset{*}{\nu}{}^i \underset{\sim}{g}_i \quad . \tag{A4}$$

For convenience in what follows we set $\theta^3 = \xi$.

The parametric equations $\theta^\alpha = 0$ $(\alpha = 1,2)$ define a material curve in space which we assume to be smooth and non-intersecting. A typical point of this curve is specified by the position vector

$$\underset{\sim}{r}(\xi,t) = \underset{\sim}{p}(0,0,\xi,t) \tag{A5}$$

and the element of arc length is given by

$$ds = (a_{33})^{\frac{1}{2}}\, d\xi \quad , \tag{A6}$$

where a_{33} is the value of g_{33} on the curve (A5). We assume that the body occupies a region of space in the neighborhood of the curve (A5) and is bounded by the surface†

$$F(\theta^1,\theta^2) = 0 \quad , \tag{A7}$$

which is such that a surface ξ = constant intersects (A7) in a closed curve $\partial\mathcal{A}$. We denote by $\mathcal{A}$ the curved section of the surface ξ = constant bounded by $\partial\mathcal{A}$. Let the points ξ_1 and ξ_2, with $\xi_1 < \xi_2$, form endpoints of a segment of the curve (A5) which we denote by $\mathcal{P}$, and designate by $\mathcal{A}_1$ and $\mathcal{A}_2$ the particular sections associated with ξ_1 and ξ_2, respectively. Now consider an arbitrary part $\mathcal{P}^*$ of the three-dimensional region occupied by the body such that: (i) $\mathcal{P}^*$ contains $\mathcal{P}$; and (ii) the boundary $\partial\mathcal{P}^*$ of $\mathcal{P}^*$ consists of the sections $\mathcal{A}_1$ and $\mathcal{A}_2$ and a portion of the surface (A7) bounded at each end by $\partial\mathcal{A}_1$ and $\partial\mathcal{A}_2$. A body so described is called a <u>rod-like body</u> and the part $\mathcal{P}^*$ forms a portion of such a body.

The mass m^* of a portion of the rod-like body is given by

---

†We could be more general and assume that the lateral bounding surface of the body is given by $F(\theta^1,\theta^2,\xi) = 0$ as in [6], but (A7) will be sufficient for the present purpose.

$$m^*(\wp^*) = \int_{\wp^*} \rho^* \, dv = \int_{\wp^*} \rho^* g^{\frac{1}{2}} \, d\theta^1 \, d\theta^2 \, d\xi \quad . \tag{A8}$$

In terms of the segment $\wp$ of the material curve (A5), the same mass has the alternative representation

$$m^* = \int_{\wp} [\int_{a} \rho^* g^{\frac{1}{2}} \, d\theta^1 \, d\theta^2] d\xi = \int_{\wp} \rho (a_{33})^{\frac{1}{2}} \, d\xi = \int_{\wp} \rho \, ds = m(\wp) \quad , \tag{A9}$$

where the density ρ per unit length of the curve (A5) is defined by

$$\lambda = \rho (a_{33})^{\frac{1}{2}} = \int_{a} \rho^* g^{\frac{1}{2}} \, d\theta^1 \, d\theta^2 \quad . \tag{A10}$$

In view of $(A3)_1$, we note that

$$\dot{\lambda} = 0 \quad . \tag{A11}$$

The curve (A5) is fixed in the rod-like body by the condition [6]

$$\int_{a} \rho^* g^{\frac{1}{2}} \theta^{\alpha} \, d\theta^1 \, d\theta^2 = 0 \quad . \tag{A12}$$

We also recall the following definitions for the resultants $\underset{\sim}{n}, \underset{\sim}{\pi}^{\alpha}, \underset{\sim}{p}^{\alpha}$:

$$\underset{\sim}{n} = \int_{a} \underset{\sim}{T}^3 \, d\theta^1 \, d\theta^2 \quad , \quad \underset{\sim}{\pi}^{\alpha} = \int_{a} \underset{\sim}{T}^{\alpha} \, d\theta^1 \, d\theta^2 \quad , \quad \underset{\sim}{p}^{\alpha} = \int_{a} \underset{\sim}{T}^3 \, \theta^{\alpha} \, d\theta^1 \, d\theta^2 \quad . \tag{A13}$$

We now assume that for the rod-like body described above, the position vector $(A1)_1$ can be approximated by

$$\underset{\sim}{p} = \underset{\sim}{r} + \theta^{\alpha} \underset{\sim}{d}_{\alpha} \quad , \quad \underset{\sim}{d}_{\alpha} = \underset{\sim}{d}_{\alpha}(\xi, t) \quad . \tag{A14}$$

Using this assumption in (A1) and (A2), we obtain

$$\begin{gathered} \underset{\sim}{v}^* = \underset{\sim}{v} + \theta^{\alpha} \underset{\sim}{w}_{\alpha} \quad , \quad \underset{\sim}{v} = \dot{\underset{\sim}{r}} \quad , \quad \underset{\sim}{w}_{\alpha} = \dot{\underset{\sim}{d}}_{\alpha} \quad , \\ \underset{\sim}{g}_{\alpha} = \underset{\sim}{d}_{\alpha} \quad , \quad \underset{\sim}{g}_3 = \underset{\sim}{a}_3 + \theta^{\alpha} \underset{\sim}{d}'_{\alpha} \quad , \quad \underset{\sim}{a}_3 = \underset{\sim}{r}' \quad , \end{gathered} \tag{A15}$$

where a prime denotes partial differentiation with respect to ξ. The equations of motion in terms of the resultants (A13) are obtained by suitable integration of $(A3)_{2,3}$ over a section $\mathcal{A}$ and are given by (for details see [6]):

$$\frac{\partial \underset{\sim}{n}}{\partial \xi} + \lambda \underset{\sim}{f} = \lambda \dot{\underset{\sim}{v}} \quad , \tag{A16}$$

$$\frac{\partial \underset{\sim}{p}^{\alpha}}{\partial \xi} + \lambda \underset{\sim}{\ell}^{\alpha} = \underset{\sim}{\pi}^{\alpha} + \lambda y^{\alpha\beta} \dot{\underset{\sim}{w}}_{\beta} \quad , \tag{A17}$$

$$\underset{\sim}{a}_3 \times \underset{\sim}{n} + \underset{\sim}{d}_{\alpha} \times \underset{\sim}{\pi}^{\alpha} + \frac{\partial \underset{\sim}{d}_{\alpha}}{\partial \xi} \times \underset{\sim}{p}^{\alpha} = \underset{\sim}{0} \quad , \tag{A18}$$

provided that

$$\lambda \underset{\sim}{f} = \int_{\mathcal{A}} \rho^* g^{\frac{1}{2}} \underset{\sim}{f}^* \, d\theta^1 \, d\theta^2 + \int_{\partial \mathcal{A}} [d\theta^2 \underset{\sim}{T}^1 - d\theta^1 \underset{\sim}{T}^2] \quad , \tag{A19}$$

$$\lambda \underset{\sim}{\ell}^{\alpha} = \int_{\mathcal{A}} \theta^{\alpha} \rho^* g^{\frac{1}{2}} \underset{\sim}{f}^* \, d\theta^1 \, d\theta^2 + \int_{\partial \mathcal{A}} \theta^{\alpha} [d\theta^2 \underset{\sim}{T}^1 - d\theta^1 \underset{\sim}{T}^2] \quad , \tag{A20}$$

and

$$\lambda y^{\alpha\beta} = \int_{\mathcal{A}} \rho^* g^{\frac{1}{2}} \theta^{\alpha} \theta^{\beta} \, d\theta^1 \, d\theta^2 \quad . \tag{A21}$$

If we adopt the approximation (A14) and identify the vectors $\underset{\sim}{d}_{\alpha}$ and the position vector $\underset{\sim}{r}$ in $(A14)_1$ with the directors in $(3.1)_2$ and the position vector $(3.1)_1$ of the curve c, then the development of this Appendix and the results given in section 3 are formally equivalent. In particular, comparison of the equations (A11) and (A16) to (A18) with those in (3.5) reveals a 1-1 correspondence between the two systems of equations provided we identify the expressions (A19) to (A21), respectively, with the assigned fields and the inertia coefficients in the theory of a directed curve discussed in section 3.

Before closing this appendix, we obtain the appropriate expression for incompressibility when the position vector is approximated by $(A14)_1$. For a three-dimensional incompressible medium, the mass density ρ^* is constant and by $(A3)_1$ the condition of incompressibility is

$$\frac{d}{dt} [\underset{\sim}{g}_1 \underset{\sim}{g}_2 \underset{\sim}{g}_3] = 0 \quad . \tag{A22}$$

Then, with the use of $(A15)_{4,5,6}$, (A22) becomes

$$\frac{d}{dt}[\mathbf{d}_1\mathbf{d}_2\mathbf{a}_3] + \theta^1 \frac{d}{dt}[\mathbf{d}_1\mathbf{d}_2 \frac{\partial \mathbf{d}_1}{\partial \xi}] + \theta^2 \frac{d}{dt}[\mathbf{d}_1\mathbf{d}_2 \frac{\partial \mathbf{d}_2}{\partial \xi}] = 0 \quad . \tag{A23}$$

In order that (A23) hold for all values of θ^1 and θ^2, we must have the separate conditions

$$\frac{d}{dt}[\mathbf{d}_1\mathbf{d}_2\mathbf{a}_3] = 0 \quad , \quad \frac{d}{dt}[\mathbf{d}_1\mathbf{d}_2 \frac{\partial \mathbf{d}_1}{\partial \xi}] = 0 \quad , \quad \frac{d}{dt}[\mathbf{d}_1\mathbf{d}_2 \frac{\partial \mathbf{d}_2}{\partial \xi}] = 0 \quad . \tag{A24}$$

On the basis of the 1-1 correspondence noted in the preceding paragraph, it is reasonable to employ the conditions (A24) as constraints in the theory of a Cosserat curve when considering incompressible media. In order to obtain some simplification of the conditions (A24), we observe that the two conditions $(A24)_{2,3}$ may alternatively be expressed in the forms

$$\begin{aligned}
&[\mathbf{w}_1\mathbf{d}_2 \frac{\partial \mathbf{d}_1}{\partial \xi}] + [\mathbf{d}_1\mathbf{w}_2 \frac{\partial \mathbf{d}_1}{\partial \xi}] + [\mathbf{d}_1\mathbf{d}_2 \frac{\partial \mathbf{w}_1}{\partial \xi}] = 0 \quad ,\\
&[\mathbf{w}_1\mathbf{d}_2 \frac{\partial \mathbf{d}_2}{\partial \xi}] + [\mathbf{d}_1\mathbf{w}_2 \frac{\partial \mathbf{d}_2}{\partial \xi}] + [\mathbf{d}_1\mathbf{d}_2 \frac{\partial \mathbf{w}_2}{\partial \xi}] = 0 \quad .
\end{aligned} \tag{A25}$$

Substituting the restricted forms (4.10) and (4.11) for $\mathbf{d}_\alpha$ and $\mathbf{w}_\alpha$ in $(A25)_{1,2}$, we see that each of the scalar triple products in the latter equations contains three coplanar vectors. Hence $(A25)_{1,2}$ are satisfied identically and incompressibility is characterized by $(A24)_1$.

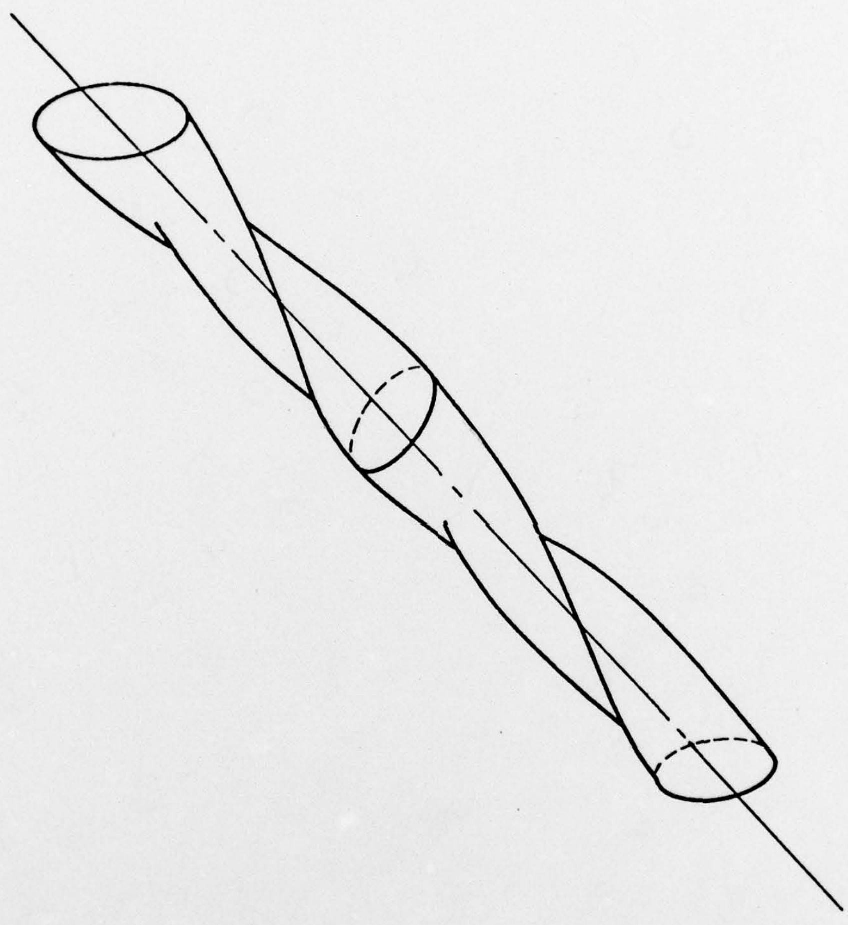


Fig. 1: An illustration indicating the steady motion of a uniformly twisted elliptical jet.

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 14 UCB/AM-77-5	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) 6 The Influence of Twist on the Motion of Straight Elliptical Jets.		5. TYPE OF REPORT & PERIOD COVERED 9 Technical Report,
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) 10 D. A. Caulk [redacted] P. M. Naghdi		8. CONTRACT OR GRANT NUMBER(s) 15 N00014-76-C-0474
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Mechanical Engineering University of California Berkeley, California 94720		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR 062-534
11. CONTROLLING OFFICE NAME AND ADDRESS Fluid Dynamics Program Office of Naval Research Arlington, VA 22217		12. REPORT DATE 11 Jun[redacted]77
		13. NUMBER OF PAGES 49 12 52p.
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report)
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Fluid jets, incompressible and inviscid fluid, direct approach, effects of gravity and surface tension, nonlinear differential equations for one-dimensional theory of a directed fluid jet, motion of a twisted elliptical jet, some special solutions, steady flow of a uniformly twisted elliptical jet.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper is concerned with a fairly detailed analysis of the motion of a straight elliptical jet of an incompressible, inviscid fluid in which the jet is allowed to twist along its axis. Our study, which includes the effects of gravity and surface tension, utilizes the nonlinear differential equations of the one-dimensional theory of a directed fluid jet. A number of theorems are proved pertaining to the motion of a twisted elliptical jet and some special solutions are obtained which illustrate the influence of twist.

DD FORM 1473, 1 JAN 73 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-014-6601

Unclassified 400 426

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)